CS 245: Extra Final Practice Problems

Disclaimer:
These may not look like the problems you'll see on the exam. They're simply ones that I've found that I believe would be good for building dexterity with the concepts. Make sure to do the provided review problems, too.

~ Good luck! :) ~

FoL Basics:

1. Building Predicates / "Definitions":

For each exercise, construct the desired predicate using the first-order logical symbols, as well as the symbols in the signature for the language, to be interpreted using the provided structure.

E.g.: Signature: $\sigma = \{\underbrace{0, 1}_{\text{consts}}, \underbrace{+, \times}_{\text{binary fn.'s}}, \underbrace{>}_{\text{binary rel.'n}}\}$,

structure: $\mathcal{M} = (\underbrace{\mathbb{N}}_{\text{domain}}, \underbrace{0, 1, +, \times, >}_{\text{all given their usual interpretations}})$

$$Prime(x) := ((x>1) \wedge \forall y \forall z ((y \times z) = x \rightarrow ((y=1) \vee (z=1))))$$

(a) Signature: $\sigma = \{0, 1, +, \times, >, \underbrace{is\_int()}_{\text{unary predicate symbol.}}\}$

structure: $\mathcal{M} = (\underbrace{\mathbb{Q}, 0, 1, +, \times, \div, >}_{\text{usual meanings}}, is\_int := \mathbb{Z})$

Even(x) := (Says that a number is even)

(b) Is_a_square(x) := (Says that a number is a perfect square)

(c) Is_not_a_multiple_of_2_or_3(x) := (Says that a number is not a multiple of 2 or 3).

(d) coprime(x, y) := (Says that two numbers are relatively prime)

Also see Tutorial 06, Practice problem 1.

FoL Semantics:

1. Finding Models for Signatures:

Honestly? Alice Gao's exercise book is a really good resource for this sort of problem, and many others in the course.

Just do all the exercises from Section 2.2 of that. Link below.

2. Proofs involving logical implication / valuations:

(a) See Section 2.3 of Gao's workbook, exercises 66-71.

(b) Show that for formulas A & B,

$$\forall x A(x) \to \forall x B(x) \not\models \forall x (A(x) \to B(x))$$

(c) Show that $\exists x \forall y A(x,y) \models \forall y \exists x A(x,y)$.

(d) Show that $\forall y \exists x A(x,y) \not\models \exists x \forall y A(x,y)$.

FoL Formal Proofs:

These problems borrowed from Alice Gao's worked examples book for the S18 edition of the course, https://cs.uwaterloo.ca/~a23gao/cs245_s18/notes/workbook_sol.pdf

1. **Exercise 100.** *Show that* $\{(\exists x\ P(x)), (\forall x\ (\forall y\ (P(x) \to Q(y))))\} \vdash (\forall y\ Q(y))$.

2. **Exercise 101.** *Show that* $\{(\exists y\ (\forall x\ P(x, y)))\} \vdash (\forall x\ (\exists y\ P(x, y)))$.

3. State and prove all 4 of the FO variants of DeMorgan's laws.

Exercise 102. *Show that* $\{(\neg(\exists x\ P(x)))\} \vdash (\forall x\ (\neg P(x)))$. *(De Morgan)*

Exercise 103. *Show that* $\{(\forall x\ (\neg P(x)))\} \vdash (\neg(\exists x\ P(x)))$. *(De Morgan)*

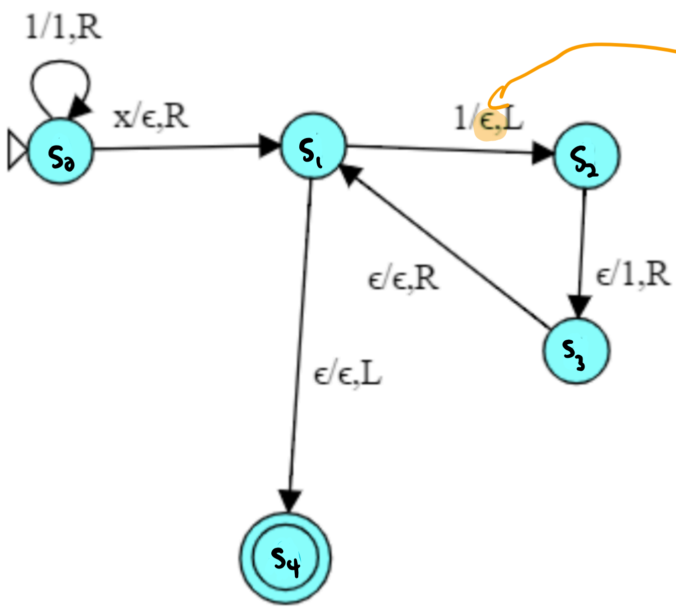
Exercise 104. *Show that* $\{(\exists x\ (\neg P(x)))\} \vdash (\neg(\forall x\ P(x)))$. *(De Morgan)*

Exercise 105. *Show that* $\{(\neg(\forall x\ P(x)))\} \vdash (\exists x\ (\neg P(x)))$. *(De Morgan)*

Turing Machines: In all examples, take $A = \{0,1\}$ to be our I/O alphabet.

1. Running TM's on input:

Consider the following TM, which has input alphabet {1,x}.



⚠ Here "ε" means blank char., B.

(A) Run the TM on the following input strings:

- 111x11
- 11x1
- x11
- 111x

(B) Describe, in words, what the TM does.

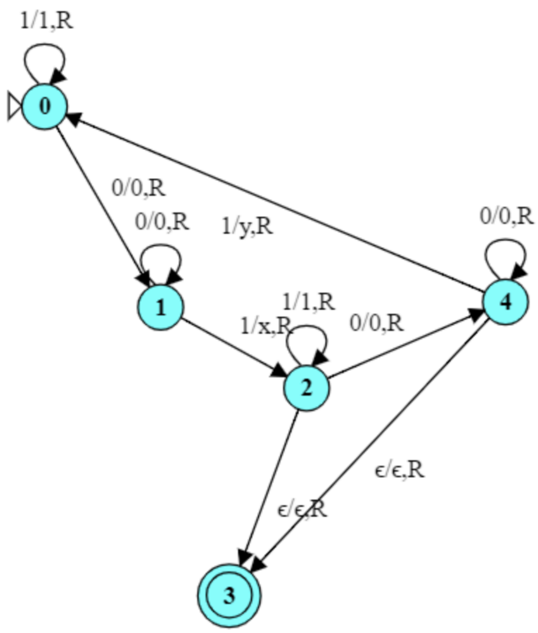
2.

These exercises borrowed from Old Dominion University's online lecture notes for CS 390, Intro to TCS, https://www.cs.odu.edu/~zeil/cs390/latest/Public/turing-jflap/index.html

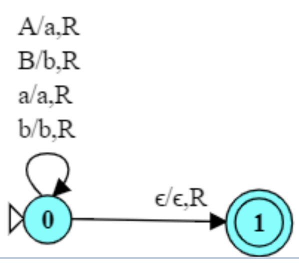
(a) Make a TM which accepting those strings containing 101 as a *substring* (I.e., the characters 1,0, and 1 need not be immediately next to each other in the string, they just have to appear somewhere in the string in that order). Generalize to an arbitrary binary string.

(b) Make a TM accepting strings ending in "101" exactly. Generalize.

3. What is the language accepted by this TM?



4. What does the following TM do? Here, the language is {a,A,b,B}.



Peano Arith., Soundness, (In)Completeness:

1. Let $\Sigma \subseteq L(\sigma)$ and $\alpha \in L(\sigma)$.
   Show that if $\Sigma \cup \{(\neg \alpha)\}$ is unsatisfiable,
   then $\Sigma \vdash \alpha$.

2. Peano Arithmetic proofs: (From Eric's notes)

Exercise. Can you prove the following theorems of Peano arithmetic? You should prove the theorems in order, and you may use the theorems you establish in the proofs of the following ones. (But you may not use later theorems to prove earlier ones to make sure you avoid circular arguments!)

1. $\Gamma_{\mathrm{PA}} \vdash \forall x(\mathbf{0}+x)=x$.
2. $\Gamma_{\mathrm{PA}} \vdash \forall x \forall y\,(S(x)+y)=S(x+y)$.
3. $\Gamma_{\mathrm{PA}} \vdash \forall x \forall y\, x+y=y+x$.
4. $\Gamma_{\mathrm{PA}} \vdash \forall x\,(\mathbf{0}\times x)=0$.
5. $\Gamma_{\mathrm{PA}} \vdash \forall x \forall y\,(S(x)\times y)=x\times y+y$.
6. $\Gamma_{\mathrm{PA}} \vdash \forall x \forall y\, x\times y=y\times x$.
7. $\Gamma_{\mathrm{PA}} \vdash \forall x \forall y \forall z\,(x\times(y+z))=((x\times y)+(x\times z))$.
8. $\Gamma_{\mathrm{PA}} \vdash \forall x \forall y \forall z\,(x+z=y+z \rightarrow x=y)$.

~ Fin. ~