CS 245: Extra Final Practice Solutions

Disclaimer:
These may not look like the problems you'll see on the exam. They're simply ones that I've found that I believe would be good for building dexterity with the concepts. Make sure to do the provided review problems, too.

~ Good luck! :) ~

FoL Basics:

1. Building Predicates / "Definitions":

For each exercise, construct the desired predicate using the first-order logical symbols, as well as the symbols in the signature for the language, to be interpreted using the provided structure.

E.g.: Signature: $\sigma = \{\underbrace{0, 1}_{\text{consts}}, \underbrace{+, \times}_{\text{binary fn.'s}}, \underbrace{>}_{\text{binary rel.'n}}\}$,

structure: $\mathcal{M} = (\underbrace{\mathbb{N}}_{\text{domain}}, \underbrace{0, 1, +, \times, >}_{\text{all given their usual interpretations}})$

$$\text{Prime}(x) := ((x > 1) \wedge \forall y \forall z ((y \times z) = x \rightarrow ((y = 1) \vee (z = 1))))$$

(a) Signature: $\sigma = \{0, 1, +, \times, >, \underbrace{\text{is\_int}()}_{\text{unary predicate symbol.}}\}$

structure: $\mathcal{M} = (\underbrace{\mathbb{Q}, 0, 1, +, \times, \div, >}_{\text{usual meanings}}, \text{is\_int} := \mathbb{Z})$

$$\text{Even}(x) := \text{is\_int}(x) \wedge \text{is\_int}(x \div (1+1))$$

(b) $\text{Is\_a\_square}(x) := \text{is\_int}(x) \wedge \exists y (\text{is\_int}(y) \wedge x = y \times y)$

(c) $\text{Is\_not\_a\_multiple\_of\_2\_or\_3}(x) :=$

$$\text{is\_int}(x) \wedge \neg [\text{Even}(x) \vee \text{is\_int}(x \div (1+1+1))]$$

(d) $\text{coprime}(x, y) :=$

$$\text{is\_int}(x) \wedge \text{is\_int}(y) \wedge \forall z [\{\text{is\_int}(z) \wedge z > 0 \wedge \text{is\_int}(x \div z) \wedge \text{is\_int}(y \div z)\} \rightarrow z = 1].$$

$\square$

Also see Tutorial 06, Practice problem 1.

FoL Semantics:

1. Finding Models for Signatures:

Honestly? Alice Gao's exercise book is a really good resource for this sort of problem, and many others in the course.

Just do all the exercises from Section 2.2 of that. Link below.

2. Proofs involving logical implication / valuations:

(a) See Section 2.3 of Gao's workbook, exercises 66-71.

(b) Show that for formulas A & B,

$$\forall x A(x) \to \forall x B(x) \not\models \forall x (A(x) \to B(x))$$

(Lu Zhongwan, p. 95)

Sol.:

To refute a logical consequence, we need only to refute a special example of it. Suppose the quasi-formulas A(x) and B(x) are atomic quasi-formulas F(x) and G(x) respectively. Then we are to prove $\forall x F(x) \to \forall x G(x) \not\models \forall x[F(x) \to G(x)]$.

Proof. Set $D = \{\alpha, \beta\}$. Form F(u) and G(u). Construct a valuation v over domain D such that $F^v = \{\alpha\}$ and $G^v = \{\beta\}$ or $\emptyset$. The

(1) $F(u)^{v(u/\alpha)} = 1;$

(2) $F(u)^{v(u/\beta)} = 0;$

(3) $G(u)^{v(u/\alpha)} = 0;$

where $G(u)^{v(u/\beta)}$ is irrelevant to the question. Then we obtain the following:

(4) $\forall x F(x)^v = 0$ (by (2));

(5) $(\forall x F(x) \to \forall x G(x))^v = 1$ (by (4));

(6) $(F(u) \to G(u))^{v(u/\alpha)} = 0$ (by (1), (3));

(7) $\forall x(F(x) \to G(x))^v = 0$ (by (6)).

By (5) and (7) we obtain

$$\forall x F(x) \to \forall x G(x) \not\models \forall x(F(x) \to G(x)).$$

$\square$

(c) Show that $\underbrace{\exists x \forall y\, A(x,y)}_{L} \models \underbrace{\forall y \exists x\, A(x,y)}_{R}$.

Sol.: Let $(\mathcal{D}, v)$ be s.t. $L^v = T$. By def. of $\models$, suffices to show that $R^v = T$.

- But $L^v = T$ means there exists some $a \in \mathcal{D}$ so that for all $b \in \mathcal{D}$, $A(x,y)^{v(x/a,\, y/b)} = T$.
- So then, in particular for all $b \in \mathcal{D}$, $A(x,y)^{v(x/a,\, y/b)} = T$

$\Rightarrow$ for all $b \in \mathcal{D}$ $(\exists x\, A(x,y))^{v(y/b)} = T$

$\Rightarrow$ $(\forall y \exists x\, A(x,y))^{v} = T$.

□

(d) Show that $\underbrace{\forall y \exists x\, A(x,y)}_{L} \not\models \underbrace{\exists x \forall y\, A(x,y)}_{R}$.

Sol.: Suffices to provide a counterexample domain & valuation for a certain choice of A.

Let $\mathcal{D} = \mathbb{R}$, and let v be the valuation on $\mathcal{D}$ assigning A to the "$<$" inequality rel'n on $\mathbb{R}$:

$$A^v(x,y) := x < y.$$

Then, certainly $v(L) = T$, since we can j. take $x := y-1$.

But $R^v = F$, b/c there is no smallest real number.

(Can j. use this as a counterexample). □

FoL Formal Proofs:

These problems borrowed from Alice Gao's worked examples book for the S18 edition of the course, https://cs.uwaterloo.ca/~a23gao/cs245_s18/notes/workbook_sol.pdf

1. **Exercise 100.** *Show that* $\{(\exists x\ P(x)), (\forall x\ (\forall y\ (P(x) \to Q(y))))\} \vdash (\forall y\ Q(y))$.

Solution: Solution 1:

1.	$(\exists x\ P(x))$	premise
2.	$(\forall x\ (\forall y\ (P(x) \to Q(y))))$	premise
3.	y_0 fresh	assumption
4.	$P(x_0)$, x_0 fresh	assumption
5.	$(\forall y\ (P(x_0) \to Q(y)))$	$\forall$e: 2
6.	$(P(x_0) \to Q(y_0))$	$\forall$e: 5
7.	$Q(y_0)$	$\to$e: 4, 6
8.	$Q(y_0)$	$\exists$e: 1, 4-7
9.	$(\forall y\ Q(y))$	$\forall$i: 3-8

Here "fresh" just means pick a fresh variable that hasn't been used. It's going to act as witness to our forall quantified.

Solution 2:

1.	$(\exists x\ P(x))$	premise
2.	$(\forall x\ (\forall y\ (P(x) \to Q(y))))$	premise
3.	$P(x_0)$, x_0 fresh	assumption
4.	y_0 fresh	assumption
5.	$(\forall y\ (P(x_0) \to Q(y)))$	$\forall$e: 2
6.	$(P(x_0) \to Q(y_0))$	$\forall$e: 5
7.	$Q(y_0)$	$\to$e: 3, 6
8.	$(\forall y\ Q(y))$	$\forall$i: 4-7
9.	$(\forall y\ Q(y))$	$\exists$e: 3-8

$\square$

2. **Exercise 101.** *Show that* $\{(\exists y\ (\forall x\ P(x, y)))\} \vdash (\forall x\ (\exists y\ P(x, y)))$.

Solution:

1.	$(\exists y\ (\forall x\ P(x, y)))$	premise
2.	$(\forall x\ P(x, y_0))$, y_0 fresh	assumption
3.	x_0 fresh	assumption
4.	$P(x_0, y_0)$	$\forall$e: 2
5.	$(\exists y\ P(x_0, y))$	$\exists$i: 4
6.	$(\forall x\ (\exists y\ P(x, y)))$	$\forall$i: 3-5
7.	$(\forall x\ (\exists y\ P(x, y)))$	$\exists$e: 1, 2-6

$\square$

3. State and prove all 4 of the FO variants of DeMorgan's laws.

Sol.:

Exercise 102. *Show that* $\{(\neg(\exists x\ P(x)))\} \vdash (\forall x\ (\neg P(x)))$. *(De Morgan)*

Solution:

1.	$(\neg(\exists x\ P(x))$	premise
2.	u fresh	assumption
3.	$P(u)$	assumption
4.	$(\exists x\ P(x))$	$\exists$i: 3
5.	$\bot$	$\bot$i: 1, 4
6.	$(\neg P(u))$	$\neg$i: 3-5
7.	$(\forall x\ (\neg P(x)))$	$\forall$i: 2-6

This is our "¬E" rule

Exercise 103. *Show that* $\{(\forall x\ (\neg P(x)))\} \vdash (\neg(\exists x\ P(x)))$. *(De Morgan)*

Solution:

1.	$(\forall x\ (\neg P(x)))$	premise
2.	$(\exists x\ P(x))$	assumption
3.	$P(u)$, u fresh	assumption
4.	$(\neg P(u))$	$\forall$e: 2
5.	$\bot$	$\bot$i: 3, 4
6.	$\bot$	$\exists$e: 2, 3-5
7.	$(\neg(\exists x\ P(x)))$	$\neg$i: 2-6

Exercise 104. *Show that* $\{(\exists x\ (\neg P(x)))\} \vdash (\neg(\forall x\ P(x)))$. *(De Morgan)*

Solution:

1.	$(\exists x\ (\neg P(x)))$	premise
2.	$(\forall x\ P(x))$	assumption
3.	$(\neg P(u))$, u fresh	assumption
4.	$P(u)$	$\forall$e: 2
5.	$\bot$	$\bot$i: 3, 4
6.	$\bot$	$\exists$e: 1, 3-5
7.	$(\neg(\forall x\ P(x)))$	$\neg$i: 2-6

Exercise 105. *Show that* $\{(\neg(\forall x\ P(x)))\} \vdash (\exists x\ (\neg P(x)))$. *(De Morgan)*

Solution:

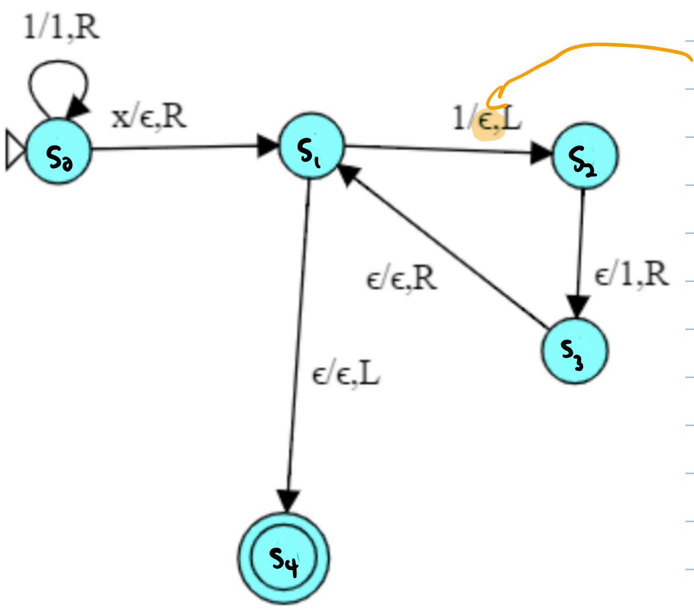
1.	$(\neg(\forall x\ P(x)))$	premise
2.	$(\neg(\exists x\ (\neg P(x))))$	assumption
3.	u fresh	assumption
4.	$(\neg P(u))$	assumption
5.	$(\exists x\ (\neg P(x)))$	$\exists$i: 4
6.	$\bot$	2, 5
7.	$P(u)$	PBC: 4-6
8.	$(\forall x\ P(x))$	$\forall$i: 3-7
9.	$\bot$	$\bot$i: 1, 8
10.	$(\exists x\ (\neg P(x)))$	PBC: 2-9

□

Turing Machines: In all examples, take $A = \{0,1\}$ to be our I/O alphabet.

1. Running TM's on input:

Consider the following TM, which has input alphabet {1,x}.



⚠ Here "ε" means blank char., B.

(A) Run the TM on the following input strings:

- 111x11
- 11x1
- x11
- 111x

(B) Describe, in words, what the TM does.

Sol.:

(A) (using config. notation)

- $111x11$: $s_0 111x11 \to 1s_0 11x11 \to 11s_0 1x11 \to 111s_0 x11 \to 111Bs_1B1 \to 111s_2BB1$
$\to 1111s_3B1 \to 1111Bs_1 1 \to 1111s_2BB \to 11111s_3B \to 11111Bs_1 \to 11111s_4B$. //

- $x11$: $s_0x11 \to Bs_1 11 \to s_2BB1 \to 1s_3B1 \to 1Bs_1 1 \to 1s_2BB \to 11s_3B$
$\to 11Bs_1 \to 11s_4B$. //

(Others are similar).

(B) This TM takes two natural numbers in base 1, and adds them. The two numbers are separated by an "x".

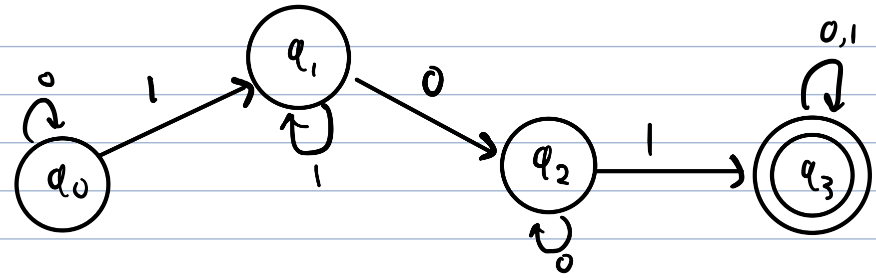
□

2.

These exercises borrowed from Old Dominion University's online lecture notes for CS 390, Intro to TCS, https://www.cs.odu.edu/~zeil/cs390/latest/Public/turing-jflap/index.html

(a) Make a TM containing 101 as a substring (i.e., not necessarily consecutively). Generalize to an arbitrary binary string.

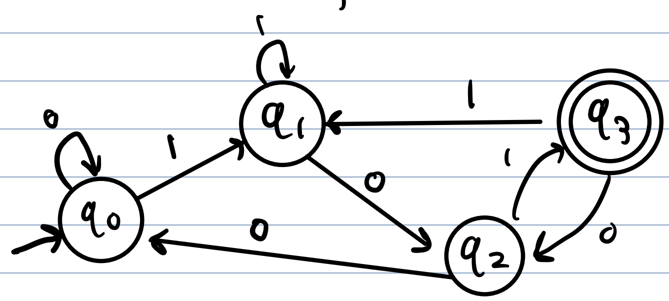
Sol.:



Write/more however you choose. The generalization is obvious. $\square$

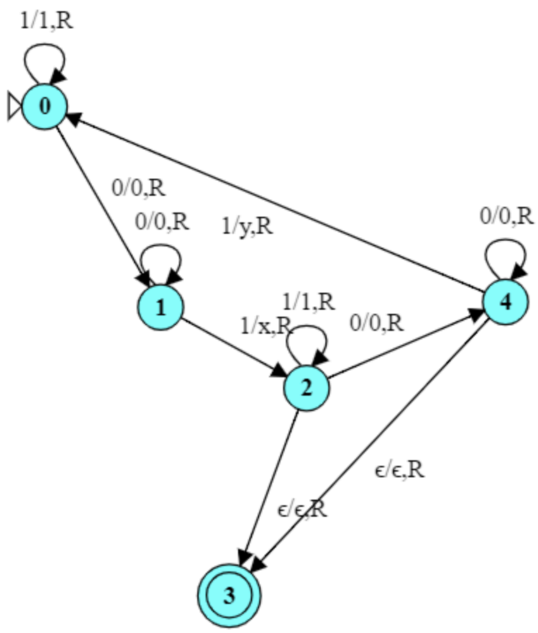
(b) Make a TM ending in 101. Generalize.

Sol.:



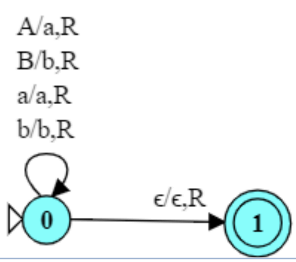
Write/more however. Again, easy to generalize. $\square$

3. What is the language accepted by this TM?



ANSWER: This TM accepts precisely those strings containing an odd number of "01"'s.

4. What does the following TM do? Here, the language is {a,A,b,B}.



ANSWER: It takes any string and converts it to all lower-case.

Peano Arith. & Gödel Incompleteness:

(1.) Let $\Sigma \subseteq L(\sigma)$ and $\alpha \in L(\sigma)$. (Gao Ex.127)

Show that if $\Sigma \cup \{(\neg\alpha)\}$ is unsatisfiable, then $\Sigma \vdash \alpha$.

Sol.: Since $\Sigma \cup \{\neg\alpha\}$ is unsatisfiable, under any valuation v s.t. $v(\Sigma) = T$, must have $v(\neg\alpha) = F$, hence $\Sigma \models \alpha$.

By FO Completeness, $\Rightarrow \Sigma \vdash \alpha$. $\square$

(2.) **Peano Arithmetic proofs:** (From Eric's notes)

Exercise. Can you prove the following theorems of Peano arithmetic? You should prove the theorems in order, and you may use the theorems you establish in the proofs of the following ones. (But you may not use later theorems to prove earlier ones to make sure you avoid circular arguments!)

1. $\Gamma_{\mathrm{PA}} \vdash \forall x(\mathbf{0}+x)=x$.
2. $\Gamma_{\mathrm{PA}} \vdash \forall x \forall y\,(S(x)+y)=S(x+y)$.
3. $\Gamma_{\mathrm{PA}} \vdash \forall x \forall y\, x+y=y+x$.
4. $\Gamma_{\mathrm{PA}} \vdash \forall x\,(\mathbf{0}\times x)=\mathbf{0}$.
5. $\Gamma_{\mathrm{PA}} \vdash \forall x \forall y\,(S(x)\times y)=x\times y+y$.
6. $\Gamma_{\mathrm{PA}} \vdash \forall x \forall y\, x\times y=y\times x$.
7. $\Gamma_{\mathrm{PA}} \vdash \forall x \forall y \forall z\,(x\times(y+z))=((x\times y)+(x\times z))$.
8. $\Gamma_{\mathrm{PA}} \vdash \forall x \forall y \forall z\,(x+z=y+z \to x=y)$.

Sol.:

Below are notes taken from the Winter 2025 edition of the course, taught by Colin Roberts, describing how to prove 3.

This is a difficult exercise, admittedly. If you can't do it formally, feel free to instead replace "|-" with "|=" in all the parts above.

21 Lecture 21 - Peano Arithmetic II

Outline

1. Commutativity of $+$
2. Associativity of $+$
3. Soundness and Completeness of Peano Arithmetic

21.1 Commutativity of $+$

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Remarks:

1. We know that $+$ in $\mathbb{N}$ is commutative.
2. But the Peano Axioms do not say this explicitly.
3. Before we can safely use the commutativity of $+$, we need to prove it!

Theorem: Addition in Peano Arithmetic is commutative; that is,

$$\varnothing \vdash_{\mathrm{PA}} \forall x \forall y(x + y \approx y + x).$$

(Notation "$\varnothing \vdash_{\mathrm{PA}}$" or "PA $\vdash$" means "provable [in *FD*] using the $\approx$ and PA axioms". For the rest of this lecture, $\vdash$ means $\vdash_{\mathrm{PA}}$.)

How can we prove this result?

We must use induction (PA7). The key first step: choose a good formula, A, for the induction property.

Choosing A(u) to be $\forall y(u + y \approx y + u)$ yields the base case

$$\varnothing \vdash \forall y(0 + y \approx y + 0)$$

and the inductive case

$$\varnothing \vdash \forall x(\forall y(x + y \approx y + x) \to \forall y(s(x) + y \approx y + s(x))).$$

Then PA7 plus one application of $(\wedge+)$ plus one application of $(\to -)$ yield the desired formula

$$\varnothing \vdash_{\mathrm{PA}} \forall x \forall y(x + y \approx y + x).$$

How should we prove $\forall y(0 + y \approx y + 0)$?

We must use induction again, to prove the base case of the main result. To control the complication, let's make it a lemma:

Lemma 21.1.1. *Peano Arithmetic has a proof of*

$$\varnothing \vdash_{\mathrm{PA}} \forall y(0 + y \approx y + 0).$$

Proof. The proof is by induction (PA7) with free variable v replacing y, and B(v) as $0+v \approx v+0$.

Basis: Prove

$$\varnothing \vdash_{PA} 0+0 \approx 0+0.$$

This is immediate from reflexivity of $\approx$.

Inductive step: Prove

$$\varnothing \vdash_{PA} \forall y\big(0+y \approx y+0 \to 0+s(y) \approx s(y)+0\big).$$

Pick v free. Assume $0+v \approx v+0$ (this is the antededent of the above implication, with the bound variable y replaced by a free v). Then, informally,

$$\begin{aligned} 0+s(v) &\approx s(0+v) && \text{PA4} \\ &\approx s(v+0) && \text{Assumption + EQSubs} \\ &\approx s(v) && \text{PA3 + EQSubs} \\ &\approx s(v)+0 && \text{PA3 + symmetry of } \approx. \end{aligned}$$

- There will be a few more of these informal arguments throughout the lecture.
- We could formalize each of them, if needed.
- We will demonstrate this, by formally proving the above informal argument here.

From the above informal proof of $0+s(v) \approx s(v)+0$, write a formal proof of

$$0+v \approx v+0 \vdash_{PA} 0+s(v) \approx s(v)+0.$$

(1)	$\varnothing$	$\vdash$	$\forall x(x+0 \approx x)$	(by [PA3])
(2)	$\varnothing$	$\vdash$	$\forall x \forall y(x+s(y) \approx s(x+y))$	(by [PA4])
(3)	$\varnothing$	$\vdash$	$\forall y(0+s(y) \approx s(0+y))$	(by ($\forall-$), (2))
(4)	$\varnothing$	$\vdash$	$0+s(v) \approx s(0+v)$	(by ($\forall-$), (3))
(5)	$\varnothing$	$\vdash$	$v+0 \approx v$	(by ($\forall-$), (1))
(6)	$\varnothing$	$\vdash$	$s(v+0) \approx s(v)$	(by EQSubs, (5))
(7)	$\varnothing$	$\vdash$	$s(v)+0 \approx s(v)$	(by ($\forall-$), (1))
(8)	$0+v \approx v+0$	$\vdash$	$0+s(v) \approx s(0+v)$	(by (+), (4))
(9)	$0+v \approx v+0$	$\vdash$	$s(v+0) \approx s(v)$	(by (+), (6))
(10)	$0+v \approx v+0$	$\vdash$	$s(v)+0 \approx s(v)$	(by (+), (7))
(11)	$0+v \approx v+0$	$\vdash$	$0+v \approx v+0$	(by ($\in$))
(12)	$0+v \approx v+0$	$\vdash$	$s(0+v) \approx s(v+0)$	(by EQSubs, (11))
(13)	$0+v \approx v+0$	$\vdash$	$s(v) \approx s(v)+0$	(by Symm of $\approx$, (10))
(14)	$0+v \approx v+0$	$\vdash$	$0+s(v) \approx s(v)+0$	(by EQTrans, (8), (12), (9), (13))

Now, complete this to a proof of

$$\varnothing \vdash \forall y(0+y \approx y+0 \to 0+s(y) \approx s(y)+0).$$

(15)	$\varnothing$	$\vdash$	$0 + v \approx v + 0 \to 0 + s(v) \approx s(v) + 0$	(by $(\to +)$, (14))
(16)	$\varnothing$	$\vdash$	$\forall y(0 + y \approx y + 0 \to 0 + s(y) \approx s(y) + 0)$	(by $(\forall+)$, (15)) (v does not occur in $\varnothing$)

Now using PA7, $(\wedge+)$ and $(\to -)$ completes the proof of the inductive step of the Lemma:

(1)	$\varnothing$	$\vdash$	$((0 + 0 = 0 + 0) \wedge \forall y((0 + y = y + 0) \to (0 + s(y) = s(y) + 0))))$ $\to \forall y(0 + y = y + 0)$	(by [PA7])
(2)	$\varnothing$	$\vdash$	$(0 + 0 = 0 + 0)$	(by Base Case)
(3)	$\varnothing$	$\vdash$	$\forall y((0 + y = y + 0) \to (0 + s(y) = s(y) + 0)))$	(by Induction Case)
(4)	$\varnothing$	$\vdash$	$((0 + 0 = 0 + 0) \wedge \forall y((0 + y = y + 0) \to (0 + s(y) = s(y) + 0))))$	(by $(\wedge+)$, (2), (3))
(5)	$\varnothing$	$\vdash$	$\forall y(0 + y = y + 0)$	(by $(\to -)$, (1), (4))

This completes the proof of the Lemma for the base case. ■

The next Lemma provides the content for the induction step of the induction case.

Lemma 21.1.2. *For each free variable* u*,*

$$\{\forall y(u + y \approx y + u)\} \vdash_{PA} \forall y(s(u) + y \approx y + s(u)).$$

Why This Suffices:

- Once we establish the above, one invocation of $(\to +)$, followed by one invocation of $(\forall+)$ will produce the desired inductive step.

Proof. Strategy: induction on free variable v in place of y, where formula B(v) is $s(u) + v \approx v + s(u)$ (keeping u free throughout).

Basis:

$$\varnothing \vdash_{PA} s(u) + 0 \approx 0 + s(u).$$

The target follows from $\forall y(0 + y \approx y + 0)$ plus $(\forall-)$ plus the symmetry of $\approx$.

Induction:

$$\varnothing \vdash_{PA} \forall y\left(s(u) + y \approx y + s(u) \to s(u) + s(y) \approx s(y) + s(u)\right).$$

Let a free v replace y. Assuming $s(u) + v \approx v + s(u)$ yields, informally,

$$\begin{aligned} s(u) + s(v) &\approx s\left(s(u) + v\right) && \text{PA4} \\ &\approx s\left(v + s(u)\right) && \text{Assumption + EQSubs} \\ &\approx s(s(v + u)) && \text{PA4 + EQSubs.} \end{aligned}$$

The premise of the Lemma implies $u + s(v) \approx s(v) + u$; thus, informally,

$$\begin{aligned} s(v) + s(u) &\approx s(s(v) + u) && \text{PA4} \\ &\approx s(u + s(v)) && \text{premise + symmetry of} \approx \text{+ EQSubs} \\ &\approx s(s(u + v)) && \text{PA4 + EQSubs.} \end{aligned}$$

The premise of the Lemma also implies $u + v \approx v + u$. Then by symmetry and transitivity of $\approx$, plus EQSubs twice, we obtain

$$s(u) + s(v) \approx s(v) + s(u),$$

so that the desired equation holds, as required.

Now applying PA7, $(\wedge+)$ and $(\rightarrow -)$ completes the proof of the lemma. ■

Now for the main proof, taking A to be $\forall y(u + y \approx y + u)$, We have

(1)	$\varnothing$	$\vdash$	$(\forall y(0 + y \approx y + 0) \wedge \forall x((\forall y(x + y \approx y + x)) \rightarrow ((\forall y(s(x) + y \approx y + s(x))))))$ $\rightarrow \forall x(\forall y(x + y \approx y + x))$	(by [PA7])
(2)	$\varnothing$	$\vdash$	$\forall y(0 + y \approx y + 0)$	(by Base Case)
(3)	$\varnothing$	$\vdash$	$\forall x((\forall y(x + y \approx y + x)) \rightarrow ((\forall y(s(x) + y \approx y + s(x)))))$	(by Induction C
(4)	$\varnothing$	$\vdash$	$(\forall y(0 + y \approx y + 0) \wedge \forall x((\forall y(x + y \approx y + x)) \rightarrow ((\forall y(s(x) + y \approx y + s(x))))))$	(by $(\wedge+)$, (2), (3
(5)	$\varnothing$	$\vdash$	$\forall x(\forall y(x + y \approx y + x))$	(by $(\rightarrow -)$, (1), (

Remarks:

1. Now that we have proved the commutativity of +, we can freely use it, when needed!
2. The other familiar properties of addition and multiplication have similar proofs.
3. One can continue: divisibility, primeness, etc.

21.2 Associativity of +

Theorem 21.2.1. *Addition (+) is associative:*

$$\varnothing \vdash_{PA} \forall x \forall y \forall z((x + y) + z \approx x + (y + z)).$$

Proof. We start by proving, for free variables u, v

$$\varnothing \vdash_{PA} \forall z((u + v) + z \approx u + (v + z)),$$

using PA7, on variable z.

<u>Basis:</u>

$$\varnothing \vdash_{PA} (u + v) + 0 \approx u + (v + 0).$$

(1)	$\varnothing$	$\vdash$	$\forall x(x + 0 \approx x)$	(by PA3)
(2)	$\varnothing$	$\vdash$	$(u + v) + 0 \approx u + v$	(by $(\forall -)$, (1))
(3)	$\varnothing$	$\vdash$	$v + 0 \approx v$	(by $(\forall -)$, (1))
(4)	$\varnothing$	$\vdash$	$u + (v + 0) \approx u + v$	(by (EQSubs), (3))
(5)	$\varnothing$	$\vdash$	$u + v \approx u + (v + 0)$	(by (Symm of $\approx$), (4))
(6)	$\varnothing$	$\vdash$	$(u + v) + 0 \approx u + (v + 0)$	(by (EQTrans), (2), (5))

<u>Induction:</u>

$$\varnothing \vdash_{PA} \forall z((u + v) + z \approx u + (v + z) \rightarrow (u + v) + s(z) \approx u + (v + s(z))).$$

~ Fin. ~