CS 245: Extra MT Practice Problems

Disclaimer:
These may not look like the problems you'll see on the exam. They're simply ones that I've found that I believe would be good for building dexterity with the concepts. Make sure to do the actual mock exam, too.

~ Good luck! :) ~

Syntax / Basics:

1. In this question, use the following proposition symbols.

p	I study for exams.
q_1	I get good grades.
q_2	I will pass the class.
r	I eat healthy food.

Translate each English sentence given into propositional logic.

(a) If I study for exams, then I get good grades.

(b) I do not eat healthy food whether or not I study for exams.

(c) I will pass the class only if I get good grades.

(d) If I do not study for exams, then I get good grades only if I eat healthy food.

(e) I will either pass the class or eat healthy food, but not both.

2. Translate each of the following sentences into formulas of the language of propositional logic. Indicate explicitly, for each proposition symbol that you define, the statement that it stands for.

(a) "Susan registered for the logic course, but Jenny did not."

(b) You get the mashed potatoes or french fries, but not both.

(c) Jenny loves opera, but she also likes BTS.

(d) She neither asserted this, nor hinted at it.

(e) Unless you give me a raise, I'll quit.

(f) Being skeptical is a necessary condition for achieving real knowledge.

(g) You are alive only if you have oxygen.

[...]

(h) A sufficient condition for Jenny to pass the logic course is that she studies and does her homework.

(i) If two lines lie in a plane, they will be parallel if and only if they neither intersect nor coincide.

(j) Oscar does not attend class unless Jenny attends.

$\square$

3. Draw a parse tree for the following formula:

$$(((p \vee q) \leftrightarrow \neg r) \wedge \neg s)$$

Structural Induction:

(1.) Prove that every $\varphi \in L^P(\sigma)$ has at least one propositional variable.

(2.) Prove that every $\varphi \in L^P(\sigma)$ has the same # of left parentheses as right parentheses.

(3.) (§2.2 Zhongwan)

2.2.3. The *degree of complexity* of $A \in Form(\mathcal{L}^p)$ is defined by recursion:

$$\begin{cases} deg(A) = 0 \text{ for atom } A. \\ deg((\neg A)) = deg(A) + 1. \\ deg((A * B)) = max(deg(A), deg(B)) + 1. \end{cases}$$

[1] Show that $deg(A) \leq$ the number of occurrences of connectives in A.

[2] Give examples of A such that $<$ or $=$ holds in [1].

(4.) Prove that in any $\varphi \in L^P(\sigma)$, each of the characters in $\{\vee, \wedge, \rightarrow, \leftrightarrow\}$ must occur w/ (string) distance at least two b/t them.

(I.e, sth like $\wedge \_ \_ \cdots \vee$, where the $\_$'s $\notin \{\vee, \wedge, \leftrightarrow, \rightarrow\}$)

Structural induction can also be applied to other sets, besides the set of formulas! For instance, consider the following “fun” exercise, taken from reddit.com/r/askmath:

(5.)

Exercise 9 Consider the set of formal strings (characters written without any implied meaning). Define the set M as follows:

- The empty string $\varepsilon \in M$. ← (base case)
- If $a, b \in M$ then $\heartsuit a \clubsuit b \in M$. ← (ind. case)

So for example, if $a = \heartsuit\clubsuit$ and $b = \heartsuit\heartsuit\clubsuit\clubsuit$ then

$$\heartsuit a \clubsuit b = \heartsuit \underbrace{\heartsuit\clubsuit}_{a} \clubsuit \underbrace{\heartsuit\heartsuit\clubsuit\clubsuit}_{b}$$

Show that for any $m \in M$, the number of $\heartsuit$'s and $\clubsuit$'s are the same.

Semantics / Argument Validity:

1. Prove, using truth tables, that

$$p \leftrightarrow q \equiv ((p \to q) \wedge (q \to p)) .$$

2. Prove, from the definitions, that

$$\varphi \equiv \psi \quad \text{iff} \quad \{\varphi\} \vDash \psi \text{ and } \{\psi\} \vDash \varphi .$$

3. Prove, from only the def'n of $\vDash$, that $\{A \to (B \wedge C)\} \vDash (A \to B) \wedge (A \to C)$.

4. Prove the following:

- $\{(A \to B) \vee (A \to C)\} \nvDash A \to (B \wedge C)$
- $\{A \to (B \vee C)\} \nvDash (A \to B) \wedge (A \to C)$

5. (Duality Theorem -- from Sp. 2025 ed. of the course).

Theorem. *(Duality)* Suppose A is a formula composed only of atoms and the connectives $\neg, \vee, \wedge$, by the formation rules concerned these three connectives. Suppose $\Delta(A)$ results from simulaneously replacing in A all occurrences of $\wedge$ with $\vee$, all occurrences of $\vee$ with $\wedge$, and each atom with its negation. Then $\neg A \equiv \Delta(A)$.

Hint: Structural Induction

6.

Translate the following argument in the language of propositional logic by using the given proposition symbols.
Determine, with proof, whether the argument is valid (sound).

Premise 1 – If knowing is a state of mind (like feeling a pain), then I could always tell by introspection whether I know.

Premise 2 – If I could always tell by introspection whether I know, then I'd never mistakenly think that I know.

Premise 3 – I sometimes mistakenly think that I know.

Conclusion – Therefore, knowing isn't a state of mind.

Define proposition symbols

p	Knowing is a state of mind.
q	I could always tell by introspection whether I know..
r	I sometimes mistakenly think that I know.

Formal Proof / Nat. Deduction:

1. Disjunctive normal form for $\longrightarrow$

   Prove, $\forall \varphi, \psi$, that $(\varphi \to \psi) \vdash (\neg\varphi \vee \psi)$

   and $(\neg\varphi \vee \psi) \vdash (\varphi \to \psi)$.

2. Distributivity of $\wedge$ over $\vee$:

   $\{(p \wedge (q \vee r))\} \vdash ((p \wedge q) \vee (p \wedge r))$

3. $\{(p \to (q \to r)), p, \neg r\} \vdash \neg q$

4. "Backwards" De Morgan's...

   - $\{(\neg\varphi \wedge \neg\psi)\} \vdash \neg(\varphi \vee \psi)$
   - $\{(\neg\varphi \vee \neg\psi)\} \vdash \neg(\varphi \wedge \psi)$

Soundness, Completeness, consistency, satisfiability...

(1.) Prove that $\forall \Gamma \subseteq L^P(\sigma)$, TFAE:

1. Γ is consistent
2. $\nexists \varphi \in L^P(\sigma)$ s.t. $\Gamma \vdash \varphi$ and $\Gamma \vdash \neg\varphi$
3. $\exists \psi \in L^P(\sigma)$ s.t. $\Gamma \nvdash \psi$.

(2.) Suppose that $\Sigma \vdash A$ and $\Gamma \vdash A$ for some sets Σ and Γ of propositional formulas and formula A. Does it follow that $\Sigma \cap \Gamma \vdash A$?

(3.) In Lemma 4 of Lecture 08 we showed that every consistent formula set is satisfiable. Now prove that every satisfiable set of formulas is consistent, thereby showing:

$$\boxed{\Gamma \text{ satisfiable} \iff \Gamma \text{ consistent}}\,.$$

(4.) Which of the following sets are consistent?

1.) $\{\varphi \wedge (\psi \to \theta),\ \varphi \to (\psi \wedge \theta),\ \neg\psi \leftrightarrow \theta\}$

2.) $\{\varphi \to \psi,\ \psi \to \theta,\ \theta \to \xi,\ \xi \to \neg\varphi\}$.

(5.) 5.2.5. Σ is said to be independent iff for each $A \in \Sigma$, $\Sigma - \{A\} \nvdash A$. Prove in propositional logic

[1] Each finite Σ has an independent $\Delta \subseteq \Sigma$ such that $\Delta \vdash A$ for all $A \in \Sigma$.

[2] Let $\Sigma = \{A_1, A_2, A_3, \cdots\}$. Find an equivalent set $\Delta = \{B_1, B_2, B_3, \cdots\}$ (that is, for all i, $\Sigma \vdash B_i$ and $\Delta \vdash A_i$) such that $B_{n+1} \vdash B_n$ but $B_n \nvdash B_{n+1}$ $(n \geq 1)$.

(From §5.2 of Lu Zhongwan)

6.

5.3.3. Suppose A contains distinct atoms $p_1, \cdots, p_n$ and t is a truth valuation. For $i = 1, \cdots, n$, let

$$A_i = \begin{cases} p_i & \text{if } p_i^t = 1, \\ \neg p_i & \text{otherwise.} \end{cases}$$

Prove

[1] $A^t = 1 \Longrightarrow A_1, \cdots, A_n \vdash A$.

[2] $A^t = 0 \Longrightarrow A_1, \cdots, A_n \vdash \neg A$.

(§5.3 Lu Zhongwan)

In mathematical logic, ***compactness*** means that if a property holds for every finite subset of an infinite set of statements, then it must hold for the entire infinite set.

This terminology comes from topology. It turns out that one can build a topological space from the set of formulas. See, e.g, https://math.stackexchange.com/questions/842/why-is-compactness-in-logic-called-compactness

Anyways, let's prove a *compactness* property for satisfiability:

7. **Theorem:** (Compactness Theorem):

$\Sigma \subseteq L^P(\sigma)$ is satisfiable $\Longleftrightarrow$ (every finite $\Sigma^0 \subseteq \Sigma$ is satisfiable.)

~ Fin. ~