CS 245: Extra MT Practice Problems — Solutions

Disclaimer:
These may not look like the problems you'll see on the exam. They're simply ones that I've found that I believe would be good for building dexterity with the concepts. Make sure to do the official practice problems, too.

Syntax / Basics:

1. In this question, use the following proposition symbols.

p	I study for exams.
q_1	I get good grades.
q_2	I will pass the class.
r	I eat healthy food.

Translate each English sentence given into propositional logic.

(a) If I study for exams, then I get good grades.
Solution: $(p \to q_1)$

(b) I do not eat healthy food whether or not I study for exams.
Solution: $((p \vee (\neg p)) \to (\neg r))$

(c) I will pass the class only if I get good grades.
Solution: $(q_2 \to q_1)$

(d) If I do not study for exams, then I get good grades only if I eat healthy food.
Solution: $((\neg p) \to (q_1 \to r))$

(e) I will either pass the class or eat healthy food, but not both.
Solution: $((q_2 \vee r) \wedge (\neg(q_2 \wedge r)))$

2. Translate each of the following sentences into formulas of the language of propositional logic. Indicate explicitly, for each proposition symbol that you define, the statement that it stands for.

(a) "Susan registered for the logic course, but Jenny did not."
Solution: Define
p Susan registered for the logic course.
q Jenny registered for the logic course.
then write $(p \wedge (\neg q))$.

(b) You get the mashed potatoes or french fries, but not both.
Solution: Define
p You get mashed potatoes.
q You get fries.
then write $((p \vee q) \wedge (\neg(p \wedge q)))$
or $((p \wedge (\neg q)) \vee ((\neg p) \wedge q))$.

(c) Jenny loves opera, but she also likes BTS.
Solution: Define
p Jenny likes opera.
q Jenny likes BTS.
then write $(p \wedge q)$.

(d) She neither asserted this, nor hinted at it.
Solution: Define
p She asserted this.
q She hinted at this.
then write $((\neg p) \wedge (\neg q))$.

(e) Unless you give me a raise, I'll quit.
Solution: Define
p You give me a raise.
q I will quit.
then write $((\neg p) \to q)$
or $(p \vee q)$.

(f) Being skeptical is a necessary condition for achieving real knowledge.
Solution: Define
p One is skeptical.
q One achieves real knowledge.
then write $(q \to p)$
or $((\neg p) \to (\neg q))$.

(g) You are alive only if you have oxygen.
Solution: Define
p You are alive.
q You have oxygen.
then write $(p \to q)$
or $((\neg q) \to (\neg p))$.

(h) A sufficient condition for Jenny to pass the logic course is that she studies and does her homework.

Solution: Define

p Jenny passes the logic course.
q Jenny studies.
r Jenny does her homework.

then write $((q \wedge r) \to p)$.

(i) If two lines lie in a plane, they will be parallel if and only if they neither intersect nor coincide.

Solution: Define

p Two lines lie in a plane.
q The lines are parallel.
r_1 The lines intersect.
r_2 The lines coincide.

then write $(p \to (q \leftrightarrow ((\neg r_1) \wedge (\neg r_2))))$.

(j) Oscar does not attend class unless Jenny attends.

Solution: Define

p Oscar attends class.
q Jenny attends class.

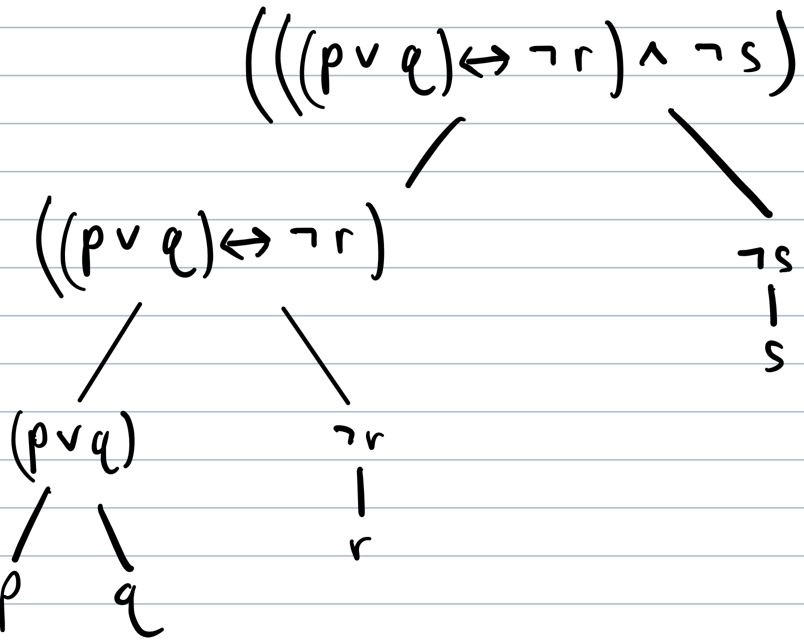
then write $((\neg p) \vee q)$
or $(p \to q)$.

$\square$

③ Draw a parse tree for the following formula:

$$(((p \vee q) \leftrightarrow \neg r) \wedge \neg s)$$

Sol.:



$\square$

Structural Induction:

(1.) Prove that every $\varphi \in L^P(\sigma)$ has at least one propositional variable.

Sol.: By structural induction on $L^P(\sigma)$.

- Base case: $\varphi \in P$: Then, φ is itself a variable.
- Ind. case:
  - ↳ $\varphi = \neg\psi$: Then, by the inductive hypothesis (IH), ψ has ≥ 1 variable, hence so does φ.
  - ↳ $\varphi = \psi \square \theta$, $\square \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$: Then by the IH, ψ, θ have ≥ 1 var, hence φ has ≥ 1 var. $\square$

(2.) Prove that every $\varphi \in L^P(\sigma)$ has the same # of left parentheses as right parentheses. $=: \mathcal{P}$

Sol.: By structural induction on $L^P(\sigma)$.

- Base case: $\varphi \in P$: Then, φ has no parentheses, so $\mathcal{P}$ is vacuously true.
- Ind. case:
  - ↳ $\varphi = \neg\psi$: Since $\mathcal{P}(\psi)$ holds by IH, also $\mathcal{P}(\neg\psi)$ holds.
  - ↳ $\varphi = \psi \square \theta$, $\square \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$: By IH, $\mathcal{P}(\psi)$ & $\mathcal{P}(\theta)$ hold. To get φ from ψ & θ, we add 1 "(" & 1 ")", so $\mathcal{P}(\varphi)$ holds. $\square$

3.

2.2.3. The *degree of complexity* of $A \in Form(\mathcal{L}^P)$ is defined by recursion:

$$\begin{cases} deg(A) = 0 \text{ for atom } A. \\ deg((\neg A)) = deg(A) + 1. \\ deg((A * B)) = max(deg(A), deg(B)) + 1. \end{cases}$$

[1] Show that $deg(A) \leq$ the number of occurrences of connectives in A. $=: \#c(\cdot)$

[2] Give examples of A such that < or = holds in [1].

(For the remaining ones, I'll omit some of the boilerplate and just sketch the reasoning for the base and inductive cases).

1.) • Base: $A \in P = Atom(A)$ implies $\#c(A) = 0$, and also $\deg A = 0$. $\therefore$ $0 \leq 0 \Rightarrow \deg A \leq \#c(A)$. ✓

• Ind.: ↳ $A = \neg B$: Then $\#c(A) = \#c(B) + 1$, and $\deg A = \deg B + 1$, so the IH $\deg B \leq \#c(B)$ implies

$$\deg B + 1 \leq \#c(B) + 1,$$

i.e. $$\deg A \leq \#c(A).$$

↳ $A = B \square D$: Now, $\#c(A) = \#c(B) + \#c(D) + 1$. And, $\deg A = \max(\deg B, \deg D) + 1$. By IH, $\deg B \leq \#c(B)$, $\deg D \leq \#c(D)$. By the natural number inequality

"$\forall a, b \in \mathbb{N}, \quad \max(a,b) \leq a + b$",

this implies $\max(\deg B, \deg D) \leq \#c(B) + \#c(D)$, so $\deg A \leq \#c(A)$.

2.) $=$: $\varphi = \neg\neg\neg\neg p$.

$<$: $\neg p \vee (p \wedge q)$. $\square$

4. Prove that in any $\varphi \in L^P(\sigma)$, each of the characters in $\{\vee, \wedge, \to, \leftrightarrow\}$ must occur w/ (string) distance at least two b/t them.

(I.e., sth like $\wedge \_ \_ \cdots \vee$, where the $\_$'s $\notin \{\vee, \wedge, \leftrightarrow, \to\}$)

__Sol.:__

- __Base:__ $\varphi = p \in P$: No such chars, so vacuously true.
- __Ind:__
  ↳ $\varphi = \neg \psi$: By IH, true for ψ, and we add __no__ additional $\vee, \wedge, \to, \leftrightarrow$ so it's true for φ.

  ↳ $\varphi = \psi \square \theta$: By IH, true for ψ, θ.

As far as accounting for the one new binary connective, note that its distance to adjacent binary connectives is minimized when we have formulas of the form:

$\varphi = (\cdots * q) \square (r \odot \cdots)$, where $q, r \in P$ & $*, \odot \in \{\wedge, \vee, \to, \leftrightarrow\}$.

By inspection, this maintains distance 2 from $\square$, on both sides. $\square$

Structural induction can also be applied to other sets, besides the set of formulas! For instance, consider the following “fun” exercise, taken from reddit.com/r/askmath:

5.

Exercise 9 Consider the set of formal strings (characters written without any implied meaning). Define the set M as follows:

- The empty string $\varepsilon \in M$. ← (base case)
- If $a, b \in M$ then $\heartsuit a \clubsuit b \in M$. ← (ind. case)

So for example, if $a = \heartsuit\clubsuit$ and $b = \heartsuit\heartsuit\clubsuit\clubsuit$ then

$$\heartsuit a \clubsuit b = \heartsuit \underbrace{\heartsuit\clubsuit}_{a} \clubsuit \underbrace{\heartsuit\heartsuit\clubsuit\clubsuit}_{b}$$

Show that for any $m \in M$, the number of $\heartsuit$'s and $\clubsuit$'s are the same.

See:
https://www.reddit.com/r/askmath/comments/1gm5dee/structural_induction/

Semantics / Argument Validity:

1. Prove, using truth tables, that

$$p \leftrightarrow q \equiv ((p \to q) \wedge (q \to p)).$$

Sol.: Below is a truth table. To see that they're equivalent, it suffices to see that they have the same values on all input combinations, which is evidently the case.

p	q	$p \leftrightarrow q$	$p \to q$	$q \to p$	$(p \to q) \wedge (q \to p)$
1	1	1*	1	1	1*
1	0	0*	0	1	0*
0	1	0*	1	0	0*
0	0	1*	1	1	1*

$\square$

2. Prove, from the definitions, that

$$\varphi \equiv \psi \quad \text{iff} \quad \{\varphi\} \vDash \psi \text{ and } \{\psi\} \vDash \varphi.$$

Sol.: $\varphi \equiv \psi :\Longleftrightarrow$ same truth value on every assignment

$\Longleftrightarrow [\forall v^*, \quad v^*(\varphi) = T \Leftrightarrow v^*(\psi) = T]$

$\Longleftrightarrow [\forall v^*, \; v^*(\varphi) = T \Rightarrow v^*(\psi) = T$ and $\forall v^*, \; v^*(\psi) = T \Rightarrow v^*(\varphi) = T]$

$\Longleftrightarrow \{\varphi\} \vDash \psi$ and $\{\psi\} \vDash \varphi$. $\square$

3. Prove, from only the def'n of $\vDash$, that $\{\underbrace{A \to (B \wedge C)}_{=:\varphi}\} \vDash \underbrace{(A \to B) \wedge (A \to C)}_{=:\psi}$.

Sol.: Let $v^*(A \to (B \wedge C)) = T$. Then:

CASES: 1. $v^*(A) = F$: Then, both $v^*(A \to B) = v^*(A \to C) = T$, since $v^*(F \to ?) = T$. So, by def.'n $v^*(\cdot \wedge \cdot)$, $v^*(\psi) = T$.

2. $v^*(A) = T$: Then, by def.'n of $v^*(\cdot \to \cdot)$, have $v^*(B \wedge C) = T$, so by $\wedge$, $v^*(B) = T$ & $v^*(C) = T$.

Since $T \to T \equiv T$, it follows that $v^*(A \to B) = v^*(A \to C) = T$, hence $v^*(\gamma) = T$. $\square$

4. Prove the following:

- $\{(A \to B) \vee (A \to C)\} \not\models A \to (B \wedge C)$
- $\{A \to (B \vee C)\} \not\models (A \to B) \wedge (A \to C)$

Sol.: In each case, it suffices to find a valuation acting as a counterexample to the definition of |=.

This means that it should satisfy the premises, but not the conclusion.

1.: $v: \begin{array}{l} A \mapsto T \\ B \mapsto F \\ C \mapsto T \end{array}$.

(In terms ofstrategy to find such a valuation, you can always use truth tables— but this is exhaustive. You might try playing around with substituting T and F into the formulas to try and find a good pick).

2.: Same v as 1. $\square$

5. (Duality Theorem -- from Sp. 2025 ed. of the course).

Theorem. *(Duality)* Suppose A is a formula composed only of atoms and the connectives $\neg, \vee, \wedge$, by the formation rules concerned these three connectives. Suppose $\Delta(A)$ results from simulaneously replacing in A all occurrences of $\wedge$ with $\vee$, all occurrences of $\vee$ with $\wedge$, and each atom with its negation. Then $\neg A \equiv \Delta(A)$.

Hint: Structural Induction

6.

Translate the following argument in the language of propositional logic by using the given proposition symbols.
Determine, with proof, whether the argument is valid (sound).
Premise 1 – If knowing is a state of mind (like feeling a pain), then I could always tell by introspection whether I know.
Premise 2 – If I could always tell by introspection whether I know, then I'd never mistakenly think that I know.
Premise 3 – I sometimes mistakenly think that I know.
Conclusion – Therefore, knowing isn't a state of mind.
Define proposition symbols

p	Knowing is a state of mind.
q	I could always tell by introspection whether I know..
r	I sometimes mistakenly think that I know.

Solution With the given notation, our premises and conclusion translate as
Premise 1: $(p \to q)$
Premise 2: $(q \to (\neg r))$
Premise 3: r
Conclusion: $(\neg p)$
We can prove that the argument is valid, for example, by using a truth table:

	p	q	r	$(p \to q)$	$(q \to (\neg r))$	$(\neg p)$
1.	1	1	1	1	0	0
2.	1	1	0	1	1	0
3.	1	0	1	0	1	0
4.	1	0	0	0	1	0
5.	0	1	1	1	0	1
6.	0	1	0	1	1	1
7.	0	0	1	1	1	1
8,	0	0	0	1	1	1

By observation, we note that in all rows where all three premises are true (in this case there is only such row, namely row 7), the conclusion is also true. This completes the proof that the argument is valid. $\square$

Formal Proof / Nat. Deduction:

The solutions are from the Fall 2017 edition of the course, taught by Alice Gao. They use a "double negation elimination" rule. Any time you see this rule, you can replace it with the sub-proof of double elimination, or form the alternative solution using RAA— it should be pretty easy. Also, they bracket things differently.

1. Disjunctive normal form for $\rightarrow$

Prove, $\forall \varphi, \psi$, that $(\varphi \rightarrow \psi) \vdash (\neg\varphi \vee \psi)$

and $(\neg\varphi \vee \psi) \vdash (\varphi \rightarrow \psi)$.

Sol.: Can substitute $\varphi = p$, $\psi = q$, iyw:

1	$(p \rightarrow q)$	premise
2	$(\neg((\neg p) \vee q))$	assumption
3	$(\neg p)$	assumption
4	$((\neg p) \vee q)$	$\vee i$: 3
5	$\perp$	$\perp i$: 2,4
6	$(\neg(\neg p))$	$\neg i$: 3–5
7	p	$\neg\neg e$: 6
8	q	$\rightarrow e$: 1,7
9	$((\neg p) \vee q)$	$\vee i$: 8
10	$\perp$	$\perp i$: 2,9
11	$(\neg(\neg((\neg p) \vee q)))$	$\neg i$: 2–10
12	$((\neg p) \vee q)$	$\neg\neg e$: 11

Other dir'n similar. $\square$

2. Distributivity of $\wedge$ over $\vee$:

$$\{(p \wedge (q \vee r))\} \vdash ((p \wedge q) \vee (p \wedge r))$$

1	$(p \wedge (q \vee r))$	premise
2	p	$\wedge e$: 1
3	$(q \vee r)$	$\wedge e$: 1
4	q	assumption
5	$(p \wedge q)$	$\wedge i$: 2,4
6	$((p \wedge q) \vee (p \wedge r))$	$\vee i$: 5
7	r	assumption
8	$(p \wedge r)$	$\wedge i$: 2,7
9	$((p \wedge q) \vee (p \wedge r))$	$\vee i$: 8
10	$((p \wedge q) \vee (p \wedge r))$	$\vee e$: 3, 4–6, 7–9

$\square$

3. $\{ (p \to (q \to r)), p, \neg r \} \vdash \neg q$

1	$(p \to (q \to r))$	premise
2	p	premise.
3	$(\neg r)$	premise
4	$(q \to r)$	$\to e: 1, 2.$
5	q	assumption
6	r	$\to e: 4, 5$
7	$\perp$	$\perp i: 5, 6.$
8	$(\neg q)$	$\neg i: 4\text{–}7$

□

4. "Backwards" De Morgan's...

$$\{ (\neg \alpha \wedge \neg \beta) \} \vdash \neg(\alpha \vee \beta)$$

Sol.:

1.	$((\neg\alpha) \wedge (\neg\beta))$	premise
2.	$(\alpha \vee \beta)$	assumption
3.	α	assumption
4.	$(\neg\alpha)$	$\wedge e: 1$
5		
6	$\perp$	$\perp i$
7	β	assumption
8	$(\neg\beta)$	$\wedge e: 1.$
9		
10	$\perp$	$\perp i$
11	$\perp$	$\vee e: 2, 3\text{-}6, 7\text{-}10$
12	$(\neg(\alpha \vee \beta))$	$\neg i: 2\text{–}11.$

□

Soundness, Completeness, consistency, satisfiability..

1. Prove that $\forall \Gamma \subseteq L^P(\sigma)$, TFAE:

1. Γ is consistent
2. $\nexists \varphi \in L^P(\sigma)$ s.t. $\Gamma \vdash \varphi$ and $\Gamma \vdash \neg\varphi$
3. $\exists \psi \in L^P(\sigma)$ s.t. $\Gamma \nvdash \psi$.

Sol.:
- $1 \Leftrightarrow 3$: Trivial; take $\psi := \bot$.
- $1 \Rightarrow 2$: AFSOC $\exists$ such a φ. Then, $\Gamma \vdash \varphi$ and $\Gamma \vdash \neg\varphi$, so $\Gamma \vdash \varphi \wedge \neg\varphi$ and $\varphi \wedge \neg\varphi \vdash \bot$, thus $\Gamma \vdash \bot$, a contradiction.
- $2 \Rightarrow 3$: Let p be any prop. variable. Then can take $\psi := p \wedge \neg p$ in 3, since if $\Gamma \vdash p \wedge \neg p$ it would prove p and $\neg p$ separately, violating 2 w/ $\varphi := p$. $\square$

2. Suppose that $\Sigma \vdash A$ and $\Gamma \vdash A$ for some sets Σ and Γ of propositional formulas and formula A. Does it follow that $\Sigma \cap \Gamma \vdash A$?

Solution: No. For example, take $A = p$, $\Sigma = \{p\}$ and $\Gamma = \{\neg\neg p\}$. Then $\Sigma \cap \Gamma = \varnothing$, but $\varnothing \nvdash p$ (this can be proved by using the fact that $\varnothing \not\models p$, and the contrapositive of the Soundness Theorem).
Remark: A tautological consequence $\Sigma \models C$ continues to hold when we add any premises to Σ. However, when we remove one or more premises from Σ, the tautological consequence may not hold anymore. $\square$

3. In Lemma 4 of Lecture 08 we showed that every consistent formula set is satisfiable. Now prove that every satisfiable set of formulas is consistent, thereby showing:

$$\boxed{\Gamma \text{ satisfiable} \iff \Gamma \text{ consistent}}.$$

<u>Sol.:</u> • Γ <u>satisfiable</u> $\Rightarrow \exists v^*: v^*(\Gamma) = T$, by definition.

• AFSOC Γ was <u>inconsistent</u>. Then, $\Gamma \vdash \bot$.

• Now, we split into cases based on $|\Gamma|$ and derive a contradiction in each.

<u>CASE 1:</u> $\Gamma = \emptyset$: Then, we have $\emptyset \vdash \bot \underset{\text{Soundness}}{\Rightarrow} \emptyset \models \bot$, which is absurd.

<u>CASE 2:</u> $\Gamma \neq \emptyset$: Then, pick some $p \in P$ appearing in one of the formulas of Γ.

• Since $\Gamma \vdash \bot$, have $\Gamma \vdash p \wedge \neg p$, so by Soundness $\Gamma \models p \wedge \neg p$, so for our v^* we have $v^*(p \wedge \neg p) = T$, which is absurd.

$\therefore \Gamma$ <u>is</u> consistent. $\square$

4. Which of the following sets are consistent?

1.) $\{ \varphi \wedge (\psi \to \theta),\ \varphi \to (\psi \wedge \theta),\ \neg\psi \leftrightarrow \theta \}$

2.) $\{ \varphi \to \psi,\ \psi \to \theta,\ \theta \to \xi,\ \xi \to \neg\varphi \}$.

<u>Sol.:</u>

1.) No. $\varphi \wedge (\psi \to \theta) \vdash \varphi$, and $\varphi, \varphi \to (\psi \wedge \theta) \vdash (\psi \wedge \theta) \vdash \psi, \theta$.

<u>But</u> we also have $\neg\psi \leftrightarrow \theta$, so one can show

$\psi, \theta, \neg\psi \leftrightarrow \theta \vdash \psi \wedge \neg\psi \vdash \perp$, hence $\Gamma \vdash \perp$

by transitivity.

2.) It depends. Note that by a previous problem, we can instead talk about satisfiability of the set of formulas in terms of satisfiability of phi, psi, xi, and theta.

If there exists a valuation so that phi is satisfied, and Gamma is satisfied, then it's inconsistent as we can deduce phi and not phi.

But if there is no such valuation, then it's consistent. □

5. 5.2.5. Σ is said to be independent iff for each $A \in \Sigma$, $\Sigma - \{A\} \nvdash A$. Prove in propositional logic

[1] Each finite Σ has an independent $\Delta \subseteq \Sigma$ such that $\Delta \vdash A$ for all $A \in \Sigma$.

[2] Let $\Sigma = \{A_1, A_2, A_3, \cdots\}$. Find an equivalent set $\Delta = \{B_1, B_2, B_3, \cdots\}$ (that is, for all i, $\Sigma \vdash B_i$ and $\Delta \vdash A_i$) such that $B_{n+1} \vdash B_n$ but $B_n \nvdash B_{n+1}$ $(n \geq 1)$.

(From §5.2 of Lu Zhongwan)

6. 5.3.3. Suppose A contains distinct atoms $p_1, \cdots, p_n$ and t is a truth valuation. For $i = 1, \cdots, n$, let

$$A_i = \begin{cases} p_i & \text{if } p_i^t = 1, \\ \neg p_i & \text{otherwise.} \end{cases}$$

Prove

[1] $A^t = 1 \Longrightarrow A_1, \cdots, A_n \vdash A$.

[2] $A^t = 0 \Longrightarrow A_1, \cdots, A_n \vdash \neg A$.

(§5.3 Lu Zhongwan)

- Here, $(\cdot)^t$ is the same as $v^*(\cdot)$, a valuation.
- Also, $1 = T$, $0 = F$.

In mathematical logic, ***compactness*** means that if a property holds for every finite subset of an infinite set of statements, then it must hold for the entire infinite set.

This terminology comes from topology. It turns out that one can build a topological space from the set of formulas. See, e.g, https://math.stackexchange.com/questions/842/why-is-compactness-in-logic-called-compactness

Anyways, let's prove a *compactness* property for satisfiability:

7. **Theorem:** (Compactness Theorem):

$$\Sigma \subseteq L^P(\sigma) \text{ is satisfiable} \iff \left(\begin{array}{c}\text{every finite } \Sigma^0 \subseteq \Sigma \\ \text{is satisfiable.}\end{array}\right)$$

Pf.: ($\Rightarrow$): Obvious.

($\Leftarrow$): Since we know satisfiable iff consistent, suffices to AFSOC that Σ is inconsistent, and derive a contradiction.

But Σ inconsistent $\Rightarrow$ $\Sigma \vdash \perp$. Any proof of $\perp$ from Σ is necessarily finite in length. Take the set of all φ appearing in this proof and call it Σ^0. Clearly $\Sigma^0 \subseteq \Sigma$ finite, but still $\Sigma^0 \vdash \perp$, $\Rightarrow$ Σ^0 inconsistent $\Rightarrow$ Σ^0 unsatisfiable, contradicting our assumption. $\square$