CS 245: TUT 103

TA name: John Hunn Smith

email: j47smith@uwaterloo.ca

website: jhs1337.github.io/cs245sp26

(Will have these notes, extra probs, etc.)

Time/Place: (Congrats, you made it!) Fridays, 2:30-3:30
MC 4042

NOTES:
- Typed solutions to these problems will be posted soon after all TUT sections are done.

UNSOLICITED ADVICE:
- ***Ask questions!
- Get help early on understanding concepts— don't wait.
- Do exercises—start simple, work up in difficulty.
    - Sources:
    - Lecture slide extra problems,
    - Lecture slide unproven theorems
    - Tutorial problems (REDO!)
    - Problems / theorems in Lu Zhongwan's book (it's a good book!)
    - Problems you think of
    - Problems people tell you
    - …

Review:

$$\sigma := \{ p, q, r, \quad p_1, p_2, p_3, \ldots, \neg, \vee, \wedge, \to, \leftrightarrow, (,) \}$$

propositional variables.

connectives

$$\sigma^n := \{ s_1 s_2 \ldots s_n : s_i \in \sigma, \text{ for all } i \}$$

$$\sigma^* := \bigcup_{n=0}^{\infty} \sigma^n .$$

$L^p(\sigma) :=$ strings in σ^* formed from finite # of applications of rules:

1.) prop. vars. $\in L^p(\sigma)$

2.) For every $\varphi \in L^p(\sigma)$, $\neg\varphi$ is also in $L^p(\sigma)$.

3.) ——— $\varphi, \psi \in L^p(\sigma)$,

$(\varphi \wedge \psi)$, $(\varphi \vee \psi)$, $(\varphi \to \psi)$, $(\varphi \leftrightarrow \psi)$ are also in $L^p(\sigma)$.

Syntax vs. **semantics**

Syntax
- Treat strings as meaningless symbols.
- Manipulate according to "grammatical" rules.

Semantics
- Strings are assigned meaning.
- Manipulations have logical (T/F) content.

Q. Why is this distinction useful?

A. Because it:
- Allows one to reuse the same argument in different contexts.
    - Different mathematical *models* satisfying the same axiomatic theory.
- Gives a descriptive language for logical thought.
- Allows one to prove/disprove that an argument itself is sound, separate from the things you are reasoning about.
- Allows logic to be treated mechanically.
    - Computers are good at syntax, but bad at semantics; humans are okay at syntax, good at semantics.

~ Problems ~

For this tutorial, all problems are taken from the Fall 2025 edition of the course, taught by Prof.'s Stephen Watt and Lila Kari.

1 Syntax of Propositional Logic

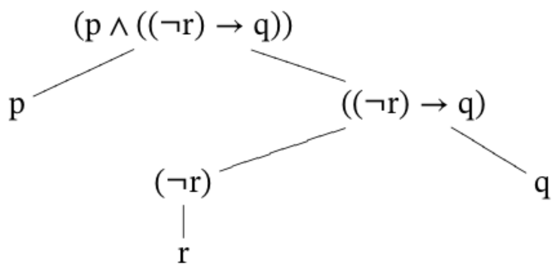
1. Given the set of connectives $C = \{\neg, \wedge, \vee, \rightarrow, \leftrightarrow\}$ and any set P of propositional symbols,
   (a) Briefly review the recursive definition of a formula.
   (b) Give some examples of formulas.
   (c) Give some examples of expressions using the given symbols that are **not** formulas.
2. Consider the propositional formula $(p \wedge (\neg r \rightarrow q))$.
   (a) Give the parse tree for the formula.
   (b) If we remove all parentheses from the given formula, then we obtain the expression $p \wedge \neg r \rightarrow q$.
   What propositional formulas (and corresponding parse trees) can one get, by adding back parentheses, in any fashion?
   (c) Which of the trees from part (b) represents the "correct" formula, according to the precedence rules for connectives?

1. (a) See above.

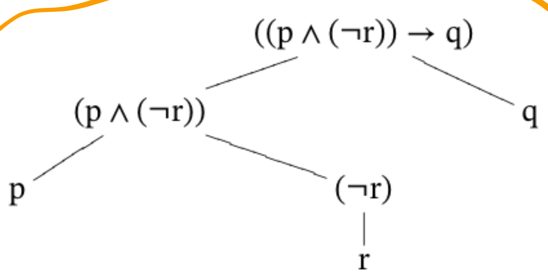
(b) p, $(p \wedge q)$, $\neg\neg(p_1 \wedge (p_2 \rightarrow p_3))$

(c) $(p \wedge)$, $(\neg p_3)$, $(p_1 \wedge p_2 \wedge p_3)$, φ.

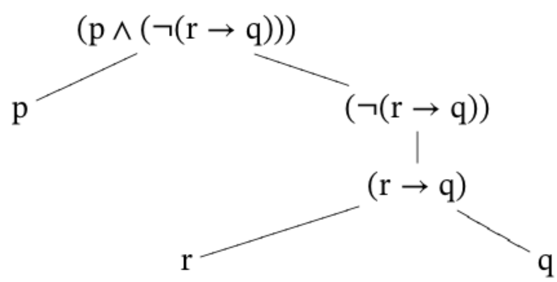
2. (a)



(b) One can check, exhaustively to see that



and



... are the only other parse trees.

(c) "Recall" $\neg > \wedge > \vee > \rightarrow > \leftrightarrow$. So, this one. //

2.

2. **Problem (Structural Induction):** Use induction on the structure of formulas to prove the following (Refer to Exercise 2.2.2 in Lu).

Let A be a formula of $Form(\mathcal{L}^P)$.
Let m_A be the number of proposition symbols in A.
Let n_A be the number of occurrences of the binary connectives $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$ in A.
Then $m_A = n_A + 1$.

Remark: The solution uses a check mark (✓) to indicate where a mark is typically assigned in the marking scheme for a similar question about structural induction on an assignment or an exam.

Recall how proofs by structural induction work...

NTS: • It holds for each base case, i.e. $\varphi \in$

$P(\varphi)$ holds, $\forall \varphi \in Atom(L^P(\sigma))$

• It holds for each inductive case, i.e.

if $\varphi = \neg\psi$ or $\varphi = \psi \square \eta$ for some $\square \in \{\wedge, \vee, \leftrightarrow, \rightarrow\}$ then

$P(\psi) \Rightarrow P(\varphi)$ or $P(\psi), P(\eta) \Rightarrow P(\psi \square \eta)$.

Pf.: **Solution:** Let R(A) be the property that $m_A = n_A + 1$ ✓ (given the above definitions for m_A and n_A).

Basis (A is p, for some proposition symbol p):

For this formula A = p, we have $m_A = 1$ and $n_A = 0$. Therefore,

$$m_A = 1 = 0 + 1 = n_A + 1, ✓$$

as required.

The induction step has two sub-cases.

A *is* $(\neg B)$, *for some formula* $B \in Form(\mathcal{L}^P)$:

The induction hypothesis is R(B), i.e. $m_B = n_B + 1$. ✓ Also, the formula has $m_A = m_B$ and $n_A = n_B$. Therefore,

$$\begin{aligned} m_A &= m_B \\ &= n_B + 1 \\ &= n_A + 1, ✓ \end{aligned}$$

as required.

A *is* $(B \star C)$, *for some formulas* $B, C \in Form(\mathcal{L}^P)$, *and some binary connective* $\star$:

The induction hypothesis is that $m_B = n_B + 1$ ✓ and $m_C = n_C + 1$. ✓ Also, we have $m_A = m_B + m_C$ and $n_A = n_B + n_C + 1$. Therefore,

$$\begin{aligned} m_A &= m_B + m_C \\ &= (n_B + 1) + (n_C + 1) \\ &= n_A + 1, ✓ \end{aligned}$$

as required. $\square$

3 Semantics of Propositional Logic

1. **Problem:** Give the complete truth table for each of the following formulas. For each formula, state whether it is a tautology, a contradiction, or satisfiable.
   (a) $(p \to (\neg p))$
   (b) $((p \to q) \wedge r)$
   (c) $((p \wedge r) \vee ((\neg r) \to q))$
   (d) the propositional formula A, which equals $((p \to (q \wedge r)) \wedge (\neg q \wedge \neg r)) \to \neg p$
2. There is an island in which certain inhabitants, called *knights*, always tell the truth, and others, called *knaves*, always lie. It is assumed that every inhabitant of this island is either a knight or a knave.
   Someone asks X: *"Are you a knight?"*
   X replies: *"If I am a knight, then I will eat my hat."*
   Prove that X will eat his hat.

First, recall the definitions of these three notions:

$\varphi \in L^P(\sigma)$ is a:

- tautology, if for any truth val. v, $v(\varphi) = T$
- satisfiable, if there exists a ——— s.t. ———.
- contradiction, if there doesn't exist a v ———.

1. (a)

p	$(\neg p)$	$(p \to (\neg p))$
1	0	0
0	1	1

- Satisfiable. Let t be: $p^t = 0$.
- NOT #, ∵ satisfiable.
- NOT tautology, ∵ $p^t = 1$ means $(p \to \neg p)^t = 0$.

(Alt.'ly, look @ rows of truth table. Each one corresponds to a truth valuation.)

(b)

p	q	r	$(p \to q)$	$((p \to q) \wedge r)$
1	1	1	1	1
1	1	0	1	0
1	0	1	0	0
1	0	0	0	0
0	1	1	1	1
0	1	0	1	0
0	0	1	1	1
0	0	0	1	0

- Satisfiable
- NOT tautology
- NOT #, ∵ satisfiable.

(c)

p	q	r	$(p \wedge r)$	$((\neg r) \to q)$	$((p \wedge r) \vee ((\neg r) \to q))$
1	1	1	1	1	1
1	1	0	0	1	1
1	0	1	1	1	1
1	0	0	0	0	0
0	1	1	0	1	1
0	1	0	0	1	1
0	0	1	0	1	1
0	0	0	0	0	0

- Satisfiable
- NOT contr., ∵ sat.
- NOT a taut.

(d)

p	q	r	$(q \wedge r)$	$(p \rightarrow (q \wedge r))$	$(\neg q \wedge \neg r)$	$(p \rightarrow (q \wedge r)) \wedge (\neg q \wedge \neg r)$	$\neg p$	A
1	1	1	1	1	0	0	0	1
1	1	0	0	0	0	0	0	1
1	0	1	0	0	0	0	0	1
1	0	0	0	0	1	0	0	1
0	1	1	1	1	0	0	1	1
0	1	0	0	1	0	0	1	1
0	0	1	0	1	0	0	1	1
0	0	0	0	1	1	1	1	1

- Tautology, $\because$ $v(\varphi) = 1, \forall v$.
- Satis., $\because$ taut.
- NOT contr., $\because$ satis.

2\.

2. There is an island in which certain inhabitants, called *knights*, always tell the truth, and others, called *knaves*, always lie. It is assumed that every inhabitant of this island is either a knight or a knave.
Someone asks X: *"Are you a knight?"*
X replies: *"If I am a knight, then I will eat my hat."*
Prove that X will eat his hat.

Solution: Define proposition symbols

p X is a knight.
q X will eat his hat.

Then X's statement translates to the formula $(p \rightarrow q)$.
Let t be a truth valuation. We prove by cases:

(a) X is a knight. Therefore $p^t = 1$ and $(p \rightarrow q)^t = 1$ (because X tells the truth). By the rule for the value of a $\rightarrow$-formula, this further implies $q^t = 1$. Thus in this case X has to eat his hat.

(b) X is a knave. Therefore $p^t = 0$ and $(p \rightarrow q)^t = 0$ (because X lies). But then by the rule for the value of a $\rightarrow$-formula, $(p \rightarrow q)^t = 1$ no matter what the value of q^t is. Hence we have reached a contradiction (because $(p \rightarrow q)^t = 0 = 1$). Therefore this case cannot occur.

Consequently, only Case **??** can hold, that is, X will eat his hat. $\square$

~ Fin. ~