CS 245: TUT 105 - Tutorial 03

Last time:
- Why logic?
- Strings, formulas, syntax vs. semantics
- Structural Induction

This time:
- Semantics: Truth tables, valuations, satisfiability

  Should've done last time (oops!)
- Logical Consequence
- Formal Proof

Review:

Semantics: Assigning meaning (T/F) to prop.'l formulae.

Let $\varphi, \psi, \theta \in L^P(\sigma)$.

(not)
Negation

φ	$\neg\varphi$
T	F
F	T

(and)
Conjunction

φ	ψ	$\varphi \wedge \psi$
T	T	T
T	F	F
F	T	F
F	F	F

(or)
Disjunction

φ	ψ	$\varphi \vee \psi$
T	T	T
T	F	T
F	T	T
F	F	F

⚠ ← NOT "exclusive or," like in English.

(if-then)
Conditional

φ	ψ	$\varphi \to \psi$
T	T	T
T	F	F
F	T	T
F	F	T

(iff)
Biconditional

φ	ψ	$\varphi \leftrightarrow \psi$
T	T	T
T	F	F
F	T	F
F	F	T

Truth Assignments:

Def'n: A truth assignment is a function

This is our set of prop.'l vars. $\mathrm{Atom}(L^P(\sigma))$

$$\nu : P \longrightarrow \{T, F\}.$$

We can extend this to a function on formulas

$\nu^* : L^P(\sigma) \longrightarrow \{T, F\}$ recursively, via:

If:
- $\varphi = \pi$, $\pi \in P$, then $\nu^*(\varphi) := \nu(\pi)$.
- $\varphi = \neg\psi$, then $\nu^*(\varphi) := \neg \nu^*(\psi)$.
- $\varphi = \psi \,\square\, \theta$, $\square \in \{\wedge, \vee, \to, \leftrightarrow\}$, then $\nu^*(\varphi) := \nu^*(\psi) \,\square\, \nu^*(\theta)$.

Truth Tables:

- Truth tables give, exhaustively, the values of all truth assignments for a given formula.
- We will see that two formulae are "the same," logically speaking, iff they have the same truth table.

Q. How many rows in a TT of a formula w/ n prop.'l vars.?

A. 2^n

In filling out a truth table for a formula, the idea is to work recursively.
Evaluate the truth values for the innermost operations, then work outwards.

E.g.: (omit... See 1.03)

Satisfiability, Tautologies, Contradictions:

Def'n: A formula is...

- ***Satisfiable*** if there exists a truth assignment making it true.
    - Equivalently, there exists a row in its TT evaluating to T.
- A ***tautology*** if every truth assignment makes it true.
    - Equivalently, every row in its TT is T.
- A ***contradiction*** if no truth assignment makes it true.
    - Equivalently, every row in its TT is F.

Logical Equivalence & Consequence:

Def'n: $\varphi, \psi \in L^P(\sigma)$ are logically equivalent if for every truth assignment v^*, $v^*(\varphi) = v^*(\psi)$. Denoted $\varphi \equiv \psi$.
or $\varphi \vDash\Dashv \psi$

- Two formulas are logically equivalent iff their truth tables are identical.

Def'n: Let $\Gamma \subseteq L^P(\sigma)$. We say φ is a logical consequence of Γ when every v^* satisfying all members of Γ also satisfies φ. Denoted by:

$$\Gamma \vDash \varphi.$$

premises conclusion

- Note that the set Γ may be empty, or infinite.

"Weird" / Counterintuitive E.g.'s:

- $\emptyset \vDash \varphi \iff \varphi$ tautology.
- $\{(p \wedge \neg p)\} \vDash \varphi$, for any φ. ← More generally, any set of premises that contains a contradiction implies any formula.
- $\{p_1\} \cup \{(p_n \to p_{n+1})\}_{n=1}^{\infty} \vDash p_k$, for any $k \in \mathbb{N}$.
  (Induction)

To prove an argument is ***valid*** (i.e. conclusion logically follows from premises):

1. Identify all atomic propositions and assign each of them a propositional variable.
2. Encode the premises and conclusion as formulas in terms of these variables.
3. Show that the conclusion is a consequence of the premises, by:
   A. Using truth tables ← SLOW!
   B. Using semantic calculus ← Ad-hoc. Also not covered.
   C. Using formal proof. ← ... both? 😓 But mechanized.

Formal Proofs:

An alternative, *syntax-based* way of proving things, relying on **Soundness** and **Completeness** theorems, which we'll prove later.

Natural Deduction System (NDS):

In NDS, a proof is a (finite) sequence of lines.
Each line has 3 elements:

- a **line number**,
- a **formula**, and
- a **justification**, consisting of a formal deduction rule and references to preceding lines in the proof to which it was applied.

Deduction Rules: • ND has 15 deduction rules.

• See the Quick Reference Guide from Eric Blais' website, next page.

Using these rules, we can say what a formal proof is...

Def'n: We say a set of premises Γ formally proves or implies φ, denoted $\Gamma \vdash \varphi$, iff there is a formal proof w/ a finite # of lines where:

- Each line is obtained by applying a deduction rule.
- All formulas used in PR lines occur in Γ
- The last line of the proof is φ, and is not contained in a subproof.

CS 245E Deduction Rules for Propositional Logic – Quick Reference Guide

PR (Premise)

$n.$	ϕ	PR

R (Re-iteration)

$m.$	ϕ	
$n.$	ϕ	R m

AS (Assumption)

$n.$	ϕ	AS

$\wedge$**I**

$k.$	ϕ	
$l.$	ψ	
$n.$	$(\phi \wedge \psi)$	$\wedge$I k, l

$\wedge$**E** (left)

$m.$	$(\phi \wedge \psi)$	
$n.$	ϕ	$\wedge$E m

$\wedge$**E** (right)

$m.$	$(\phi \wedge \psi)$	
$n.$	ψ	$\wedge$E m

$\vee$**I** (left)

$m.$	ϕ	
$n.$	$(\phi \vee \psi)$	$\vee$I m

$\vee$**I** (right)

$m.$	ψ	
$n.$	$(\phi \vee \psi)$	$\vee$I m

$\vee$**E**

$i.$	$(\phi \vee \psi)$	
$j.$	ϕ	AS
$k.$	θ	
$l.$	ψ	AS
$m.$	θ	
$n.$	θ	$\vee$E i, j–k, l–m

$\to$**I**

$k.$	ϕ	AS
$l.$	ψ	
$n.$	$(\phi \to \psi)$	$\to$I k–l

$\to$**E**

$k.$	$(\phi \to \psi)$	
$l.$	ϕ	
$n.$	ψ	$\to$E k, l

$\leftrightarrow$**I**

$k.$	$(\phi \to \psi)$	
$l.$	$(\psi \to \phi)$	
$n.$	$(\phi \leftrightarrow \psi)$	$\leftrightarrow$I k, l

$\leftrightarrow$**E** (forward)

$m.$	$(\phi \leftrightarrow \psi)$	
$n.$	$(\phi \to \psi)$	$\leftrightarrow$E m

$\leftrightarrow$**E** (reverse)

$m.$	$(\phi \leftrightarrow \psi)$	
$n.$	$(\psi \to \phi)$	$\leftrightarrow$E m

$\neg$**I**

$k.$	ϕ	AS
$l.$	$\perp$	
$n.$	$\neg\phi$	$\neg$I k–l

$\neg$**E**

$k.$	ϕ	
$l.$	$\neg\phi$	
$n.$	$\perp$	$\neg$E k, l

Reductio **A**d **A**bsurdum

$k.$	$\neg\phi$	AS
$l.$	$\perp$	
$n.$	ϕ	RAA k–l

$\perp$**E**

$m.$	$\perp$	
$n.$	ϕ	$\perp$E m

— End Review ~

Practice Problems:

1. Semantics:

(a.) Form TT for $((p \to q) \wedge r)$. Decide if it's satisfiable, tautology, contradiction.

Sol'n:

p	q	r	$(p \to q)$	$((p \to q) \wedge r)$
1	1	1	1	1
1	1	0	1	0
1	0	1	0	0
1	0	0	0	0
0	1	1	1	1
0	1	0	1	0
0	0	1	1	1
0	0	0	1	0

- Satisfiable
- NOT tautology
- NOT #, ∵ satisfiable.

(b.) BONUS: (From Sp. 25 ed. of course)

Valid and Invalid Arguments

1. Translate the following argument in the language of propositional logic by using the given proposition symbols.
   Determine, with proof, whether the argument is valid (sound).
   Premise 1 – If knowing is a state of mind (like feeling a pain), then I could always tell by introspection whether I know.
   Premise 2 – If I could always tell by introspection whether I know, then I'd never mistakenly think that I know.
   Premise 3 – I sometimes mistakenly think that I know.
   Conclusion – Therefore, knowing isn't a state of mind.
   Define proposition symbols
   p Knowing is a state of mind.
   q I could always tell by introspection whether I know..
   r I sometimes mistakenly think that I know.

   Solution With the given notation, our premises and conclusion translate as
   Premise 1: $(p \to q)$
   Premise 2: $(q \to (\neg r))$
   Premise 3: r
   Conclusion: $(\neg p)$
   We can prove that the argument is valid, for example, by using a truth table:

	p	q	r	$(p \to q)$	$(q \to (\neg r))$	$(\neg p)$
1.	1	1	1	1	0	0
2.	1	1	0	1	1	0
3.	1	0	1	0	1	0
4.	1	0	0	0	1	0
5.	0	1	1	1	0	1
6.	0	1	0	1	1	1
7.	0	0	1	1	1	1
8,	0	0	0	1	1	1

By observation, we note that in all rows where all three premises are true (in this case there is only such row, namely row 7), the conclusion is also true. This completes the proof that the argument is valid. ☐

② Logical Consequence:

Show that for all $j \in \mathbb{N}$,

$$\Gamma := \{(p_i \to p_{i+1}) : i \in \mathbb{N}\} \vDash \underbrace{(p_j \to p_{j+2})}_{=:\varphi_j}.$$

Sol'n:

- Fix $j \in \mathbb{N}$. We're showing $\Gamma \vDash \varphi_j$.
- Let v^* be any assignment s.t. $\forall i \in \mathbb{N}$, $v^*((p_i \to p_{i+1})) = T$.
- CASES:

1.) $v^*(p_j) = F$: Then, trivially from the valuation of $\to$,

$v^*((p_j \to p_{j+2})) = T$, regardless of $v^*(p_{j+2})$'s value. ✓

2.) $v^*(p_j) = T$: Then, since $v^*((p_j \to p_{j+1})) = T$,

$v^*(p_{j+1}) = T$ as well. (Otherwise, we'd have T —> F, which evaluates to F.)

Similarly, $v^*(p_{j+1} \to p_{j+2}) = T$ implies $v^*(p_{j+2}) = T$.

Thus, $T \equiv T \to T \equiv v^*(p_j) \longrightarrow v^*(p_{j+2})$

$\equiv v^*((p_j \to p_{j+2}))$, as needed. $\square$

3. Formal Proof:

(a): Commutativity of $\wedge$: $\{(p \wedge q)\} \vdash (q \wedge p)$

Sol.:

1.	$(p \wedge q)$	PR	
2.	p	$\wedge$E	1
3.	q	$\wedge$E	1
4.	$(q \wedge p)$	$\wedge$I	3, 2

$\square$

(b): Associativity of $\wedge$: $\{((p \wedge q) \wedge r)\} \longrightarrow (p \wedge (q \wedge r))$

Sol.:

1.	$((p \wedge q) \wedge r)$	PR	
2.	$(p \wedge q)$	$\wedge$E	1
3.	r	$\wedge$E	1
4.	p	$\wedge$E	2
5.	q	$\wedge$E	2
6.	$(q \wedge r)$	$\wedge$I	5, 3
7.	$(p \wedge (q \wedge r))$	$\wedge$I	4, 6

$\square$

(c): Commutativity of $\vee$: $\{(p \vee q)\} \vdash (q \vee p)$

Sol.:

1.	$(p \vee q)$	PR	
2.	p	AS	
3.	$(q \vee p)$	$\vee$I	2
4.	q	AS	
5.	$(q \vee p)$	$\vee$I	4
6.	$(q \vee p)$	$\vee$E	1, 2-3, 4-5.

$\square$

(d): Transitivity of $\rightarrow$: $\{(p \rightarrow q), (q \rightarrow r)\} \vdash \{(p \rightarrow r)\}$

Sol.:

1.	$(p \rightarrow q)$	PR	
2.	$(q \rightarrow r)$	PR	
3.	p	AS	
4.	q	$\rightarrow$E	1, 3
5.	r	$\rightarrow$E	2, 4
6.	$(p \rightarrow r)$	$\rightarrow$I	3, 5

$\square$

~ Fin. ~