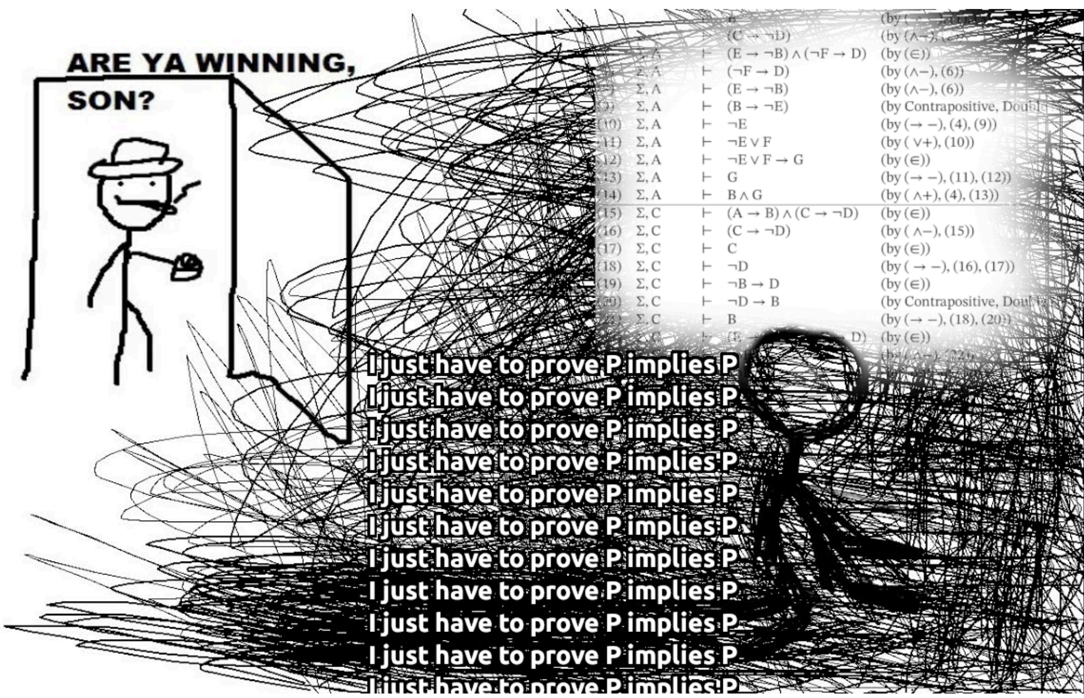
CS 245: TUT 105 - Tutorial 04

Last time:
- Semantics: Truth tables, valuations, satisfiability
- Logical Consequence
- Syntax: Formal Proof intro

This time:
- More examples of formal proofs
- Soundness & Completeness

Meme of the week:



CS 245E Deduction Rules for Propositional Logic – Quick Reference Guide

PR (Premise)

$n.$	ϕ	PR

R (Re-iteration)

$m.$	ϕ	
$n.$	ϕ	R m

AS (Assumption)

$n.$	ϕ	AS

$\wedge$**I**

$k.$	ϕ	
$l.$	ψ	
$n.$	$(\phi \wedge \psi)$	$\wedge$I k, l

$\wedge$**E** (left)

$m.$	$(\phi \wedge \psi)$	
$n.$	ϕ	$\wedge$E m

$\wedge$**E** (right)

$m.$	$(\phi \wedge \psi)$	
$n.$	ψ	$\wedge$E m

$\vee$**I** (left)

$m.$	ϕ	
$n.$	$(\phi \vee \psi)$	$\vee$I m

$\vee$**I** (right)

$m.$	ψ	
$n.$	$(\phi \vee \psi)$	$\vee$I m

$\vee$**E**

$i.$	$(\phi \vee \psi)$	
$j.$	ϕ	AS
$k.$	θ	
$l.$	ψ	AS
$m.$	θ	
$n.$	θ	$\vee$E i, j–k, l–m

$\rightarrow$**I**

$k.$	ϕ	AS
$l.$	ψ	
$n.$	$(\phi \rightarrow \psi)$	$\rightarrow$I k–l

$\rightarrow$**E**

$k.$	$(\phi \rightarrow \psi)$	
$l.$	ϕ	
$n.$	ψ	$\rightarrow$E k, l

$\leftrightarrow$**I**

$k.$	$(\phi \rightarrow \psi)$	
$l.$	$(\psi \rightarrow \phi)$	
$n.$	$(\phi \leftrightarrow \psi)$	$\leftrightarrow$I k, l

$\leftrightarrow$**E** (forward)

$m.$	$(\phi \leftrightarrow \psi)$	
$n.$	$(\phi \rightarrow \psi)$	$\leftrightarrow$E m

$\leftrightarrow$**E** (reverse)

$m.$	$(\phi \leftrightarrow \psi)$	
$n.$	$(\psi \rightarrow \phi)$	$\leftrightarrow$E m

$\neg$**I**

$k.$	ϕ	AS
$l.$	$\bot$	
$n.$	$\neg\phi$	$\neg$I k–l

$\neg$**E**

$k.$	ϕ	
$l.$	$\neg\phi$	
$n.$	$\bot$	$\neg$E k, l

Reductio **A**d **A**bsurdum

$k.$	$\neg\phi$	AS
$l.$	$\bot$	
$n.$	ϕ	RAA k–l

$\bot$**E**

$m.$	$\bot$	
$n.$	ϕ	$\bot$E m

Review:

Last time, we introduced the Natural Deduction proof system, along with its 15 rules of inference. They're displayed on the previous page.

A ***proof*** in this system is defined to be a finite sequences of lines of the form:

$$N. \quad \psi \quad \text{Rule } ?, ? \ldots ?$$

where $N.$ is line #, $\psi \in L^P(\sigma)$, and...
..."Rule" is one of the 15 deduction rules applied to previous lines "?,?,...,?" appearing in the proof.

For $\Gamma \subseteq L^P(\sigma)$ and $\varphi \in L^P(\sigma)$, we say $\Gamma \vdash \varphi$, "Γ proves φ," if there is a formal proof whose last line is φ, w/ all "PR" (premise) lines coming from Γ and no earlier line in the proof contains φ. ← minimality condition

Eg's of formal proofs: (See L.06 notes!)

You also learned about some common "Derived Rules" of logic, which can be proven from the fundamental rules of ND:

- **Modus Tollens**, aka "denying the consequent":

$$\{p \to q, \neg q\} \vdash \neg p$$

- **Law of the Excluded Middle:**

$$\vdash (p \vee \neg p)$$

- **Disjunctive Syllogism:**

$$\{(p \vee q), \neg p\} \vdash q$$

- **DeMorgan's Laws:**

$$\neg(p \vee q) \vdash (\neg p \wedge \neg q) \quad \text{and} \quad \neg(p \wedge q) \vdash (\neg p \vee \neg q)$$

Aside: Some of you may have heard about a notion of "DeMorgan's Laws" in elementary set theory…

- If $A, B \subseteq X$ for some universe set X, then:

$$(A \cup B)^c = A^c \cap B^c, \qquad (A \cap B)^c = A^c \cup B^c,$$

where $A^c = X \setminus A$ is the set complement of A in X.

- More generally, for any family $\{A_i\}_{i \in I}$ of sets $A_i \subseteq X$,

$$\left(\bigcup_{i \in I} A_i\right)^c = \bigcap_{i \in I} A_i^c \quad \text{and} \quad \left(\bigcap_{i \in I} A_i\right)^c = \bigcup_{i \in I} A_i^c .$$

There is a nice interplay between the DeMorgan's Laws of set theory and the DeMorgan's Laws of propositional logic; they're basically the same.

The duality stems from the definition of intersection and union, which involve conjunction and disjunction:

$$A \cup B := \{x \in X : x \in A \vee x \in B\}$$

$$A \cap B := \{x \in X : x \in A \wedge x \in B\}$$

The proof of set theory DeMorgan's therefore follows directly from these definitions and the DeMorgan's of logic.

Exercises:

1. Try to prove the 4 Derived Laws above, all by yourself.
2. Prove the (finite, 2-argument) DeMorgan's laws of set theory using DeMorgan's laws for logic.

Soundness & Completeness:

We've learned two different notions of logical consequence / "analyzing arguments":

$$\underset{\text{semantic}}{\Gamma \vDash \varphi} \quad \text{and} \quad \underset{\text{syntactic}}{\Gamma \vdash \varphi}.$$

It would be nice if these two notions are equivalent to each other; i.e., a set of premises logically implies a conclusion iff there exists a formal proof of the conclusion from the premises.

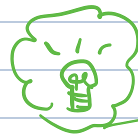
But we cannot expect this to be true without proving it.

Soundness: Whenever we obtain a proof of φ from Γ, we know that φ must also follow from Γ logically.

Thm.: (Soundness of prop.'l logic)

For every $\Gamma \subseteq L^P(\sigma)$ and $\varphi \in L^P(\sigma)$,

$$\Gamma \vdash \varphi \Rightarrow \Gamma \vDash \varphi.$$



You should always try and find such interpretations when you meet a new theorem. Also ask, "Why do I care? What does this result give us / allow us to do?

Intuitively:
"Our system of propositional logic + Natural Deduction cannot prove false statements."

The proof of the Soundness theorem uses a couple of basic facts about logical consequence:

Prop. 1:

If $\Gamma \vDash \varphi$ then also $\Gamma' \vDash \varphi$, $\forall \Gamma' \subseteq L^P(\sigma)$ s.t. $\Gamma' \supseteq \Gamma$.

Prop. 2:

Let $\Gamma \subseteq L^P(\sigma)$ and $\varphi_1, \ldots, \varphi_k \in L^P(\sigma)$ such that $\Gamma \vDash \varphi_1, \ldots, \Gamma \vDash \varphi_k$.

If $\psi \in L^P(\sigma)$ is s.t. $\{\varphi_1, \ldots, \varphi_k\} \vDash \psi$, then $\Gamma \vDash \psi$.

Prop 3: Semantic forms of all 15 rules of natural deduction.
(Omitted. See notes!)

Completeness: Any time we have a logically valid argument of φ from Γ, we should be able to find a proof of φ from Γ.

Def'n: $\Gamma \subseteq L^P(\sigma)$ is:

- **consistent** when $\Gamma \nvdash \perp$
- **inconsistent** when $\Gamma \vdash \perp$, and
- **maximally consistent** when Γ is consistent and, for any $\varphi \notin \Gamma$, $\Gamma \cup \{\varphi\}$ is inconsistent.

Using these, we prove a few basic facts needed to prove the Completeness Theorem.

For instance, that consistent sets of formulas are satisfiable.

— (Other results omitted! See notes.) —

Thm.: (Completeness of prop.'l logic):
For every $\Gamma \subseteq L^P(\sigma)$ and $\varphi \in L^P(\sigma)$, $\Gamma \models \varphi \Rightarrow \Gamma \vdash \varphi$.

Intuitively,
"Whenever something is true in proposition logic, there exists a proof for it in ND."

Thus, $\boxed{\Gamma \vdash \varphi \Leftrightarrow \Gamma \models \varphi}$ ... very satisfying. :)

— End Review. —

Practice Problems:

1. Formal Proofs:

(a.) Double negation:

$\{p\} \vdash \neg\neg p$

Sol.:

1.	p	PR	
2.	$\neg p$	AS	
3.	$\perp$	$\neg$E	1, 2
4.	$\neg\neg p$	$\neg$I	2-3

//

$\{\neg\neg p\} \vdash p$

Sol.:

1.	$\neg p$	AS	
2.	$\neg\neg p$	PR	
3.	$\perp$	$\neg$E	1, 2
4.	p	RAA	1-3

← Notice the order!

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(b). Excluded middle: $\vdash (p \vee \neg p)$

Sol.:

1.	$\neg(p \vee \neg p)$	AS	
2.	p	AS	
3.	$(p \vee \neg p)$	$\vee$I (right)	2
4.	$\perp$	$\neg$E	3, 1
5.	$\neg p$	AS	
6	$(p \vee \neg p)$	$\vee$I (left)	5
7.	$\perp$	$\neg$E	6, 1
8.	$\perp$	$\vee$E	1, 2-4, 5-7
9.	$(p \vee \neg p)$	RAA	1-8

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⚠ This line is missing from the proof in Eric Blais' notes!

(c). De Morgan's II: (Not in the notes): $\{\neg(p \wedge q)\} \vdash (\neg p \vee \neg q)$

Sol.:

1.	$\neg(p \wedge q)$	PR	
2.	$\neg(\neg p \vee \neg q)$	AS	
3.	$\neg p$	AS	
4.	$\neg p \vee \neg q$	$\vee$I (right)	3
5.	$\perp$	$\neg$E	4, 2
6.	$\neg q$	AS	
7.	$\neg p \vee \neg q$	$\vee$I (left)	6
8.	$\perp$	$\neg$E	7, 2
9.	p	RAA	3-5
10	q	RAA	6-8

11.	$p \wedge q$	$\wedge I$	9,10
12.	$\bot$	$\neg E$	11,1
13.	$(\neg p \vee \neg q)$	RAA	2-12 //

2. Soundness:

Show that there is no Natural Deduction proof to witness

$$\{p \to q,\ r \to q\} \vdash \{p \to r\}.$$

Sol.: By Soundness Thm., $\Gamma \vdash \varphi \Rightarrow \Gamma \models \varphi$.

Taking the contrapositive, $\Gamma \not\models \varphi \Rightarrow \Gamma \nvdash \varphi$.

We will show $\Gamma \not\models \varphi$ by exhibiting a valuation v:

$v^*(\Gamma) = T$ but $v^*(\varphi) = F$:

Let: $v: \begin{array}{l} p \mapsto T \\ q \mapsto T \\ r \mapsto F \end{array}$. Then, $\begin{array}{l} v^*(p \to q) = T \\ v^*(r \to q) = T, \end{array}$

but $v^*(p \to r) = F$, so this v^* shows $\Gamma \not\models \varphi$, hence $\Gamma \nvdash \varphi$. $\square$

3. Soundness & Completeness:

Suppose that $\Sigma \vdash A$ and $\Sigma \cup \{A\} \vdash B$, for some set Σ and propositional formulas A and B. Is it always the case that $\Sigma \vdash B$? Provide a proof or a counterexample.

Sol.: It is always the case. **Proof:**

By Soundness, $\Sigma \vdash A$ and $\Sigma \cup \{A\} \vdash B$ imply $\Sigma \models A$ and $\Sigma \cup \{A\} \models B$, respectively. Let v^* be any truth val.'n s.t. $v^*(\Sigma) = T$. Since $\Sigma \models A$, $v^*(A) = T$.

Hence, $v^*(\Sigma \cup \{A\}) = T$. And since $\Sigma \cup \{A\} \models B$, also $v^*(B) = T$. Thus, we've shown that $\Sigma \models B$.

By Completeness, this means $\Sigma \vdash B$. $\square$

— Fin. —