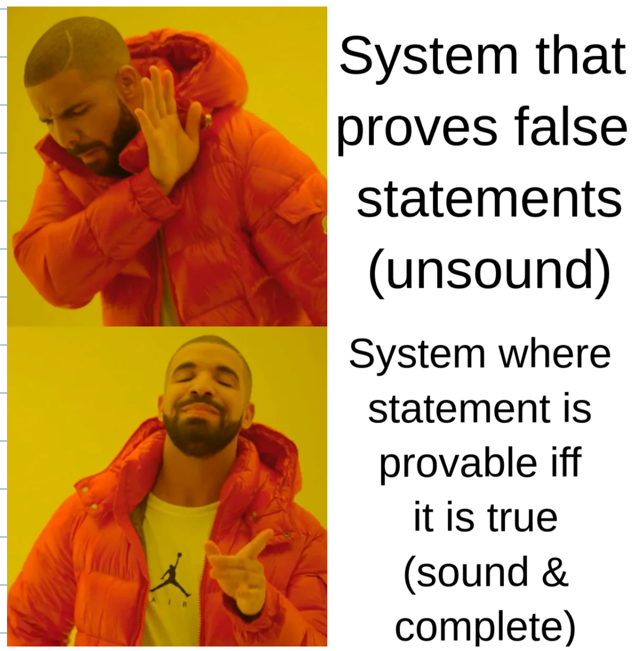
CS 245: TUT 105 - Tutorial 05

Midterm Review

NOTE:

- Lecture 08 is not covered by Assignment 02, but it is covered on the midterm!
- I will post some extra review problems later today or tomorrow. Solutions will be provided a bit later.

Meme(s) of the week:



Review:

Today, we will focus on reviewing some Completeness concepts and doing practice problems.

For general exam review, I would carefully revise from Eric Blais' notes, and do the proofs from there or exercises in Lu Zhongwan's book. Also, practice problems, when they become available.

Defi'n: $\Gamma \subseteq L^P(\sigma)$ is:

- **consistent** when $\Gamma \nvdash \perp$
- **inconsistent** when $\Gamma \vdash \perp$, and
- **maximally consistent** when Γ is consistent and, for any $\varphi \notin \Gamma$, $\Gamma \cup \{\varphi\}$ is inconsistent.

E.g.:

$\{(p \wedge q), \neg p\}$, $\{(p \to q), p, \neg q\}$, $\{(p \vee q), \neg p, \neg q\}$, $\{(p \vee q), (\neg p \vee r), \neg q, \neg r\}$ are all inconsistent.

Exercise: Prove it, formally.

E.g.: If $v : P \to \{T, F\}$ is any truth assignment to your prop.'l variables, then the set of formulas

$$\mathcal{T}_v = \{\psi \in L^P(\sigma) : v^*(\psi) = T\}$$

is maximally consistent.

Exercise: Prove it!

Hint: Use completeness Thm.

Prop.: Let $\Gamma \subseteq L^P(\sigma)$ be maximally consistent.

If $\varphi_1, \varphi_2 \in \Gamma$, then so is $(\varphi_1 \wedge \varphi_2)$.

Pf.: Since $\varphi_1, \varphi_2 \in \Gamma$, $\Gamma \vdash (\varphi_1 \wedge \varphi_2)$. By equivalence of (1.) $\Leftrightarrow$ (3.) in L.08, Proposition 1, have $(\varphi_1 \wedge \varphi_2) \in \Gamma$. $\square$

$\varphi \in \Gamma \Leftrightarrow \Gamma \vdash \varphi$

Proof Sketch of Completeness Thm:

- Claim 1: To show Completeness, it suffices to show that every consistent set is satisfiable (Lemma 4).

Reason:

Let $\Gamma \vDash \varphi$. Then $\forall v^*$, $v^*(\Gamma) = T \Rightarrow v^*(\varphi) = T$.

Equivalently, $\nexists v^*$ s.t. $v^*(\Gamma) = T$ and $v^*(\varphi) = F$.

Equivalently, $\nexists v^*$ s.t. $v^*(\Gamma \cup \{\neg\varphi\}) = T$.

Thus $\Gamma \cup \{\neg\varphi\} \vDash \bot$.

- By the contrapos. of Lemma 4, $\Gamma \cup \{\neg\varphi\} \vdash \bot$.
- Use:

k.	$\neg\varphi$	AS
	$\vdots$	
l.	$\bot$	
n.	φ	RAA k-l

... to get $\Gamma \vdash \varphi$, as needed. $\square$

Now, to show every consistent set is satisfiable, we:

1. First show every consistent set can be expanded to a maximally consistent set.
   A. (Greedily add formulas that don't introduce contradictions, one at a time.)
2. Show every maximally consistent set is satisfiable
   A. (Lemma 2)
3. Reduce from the maximally consistent set back down to the original set
   A. (Proposition 3)

~ ADVICE: Know the proofs of Soundness and Completeness, and the related Propositions! ~

End Review.

Practice Problems:

(1.) Soundness & Completeness (from last week)

Suppose that $\Sigma \vdash A$ and $\Sigma \cup \{A\} \vdash B$, for some set Σ and propositional formulas A and B. Is it always the case that $\Sigma \vdash B$? Provide a proof or a counterexample.

Sol.: It is always the case. Proof:

By Soundness, $\Sigma \vdash A$ and $\Sigma \cup \{A\} \vdash B$ imply $\Sigma \models A$ and $\Sigma \cup \{A\} \models B$, respectively. Let v^* be any truth val.'n s.t. $v^*(\Sigma) = T$. Since $\Sigma \models A$, $v^*(A) = T$. Hence, $v^*(\Sigma \cup \{A\}) = T$. And since $\Sigma \cup \{A\} \models B$, also $v^*(B) = T$. Thus, we've shown that $\Sigma \models B$. By Completeness, this means $\Sigma \vdash B$. □

(2.) Formal Proof: $\{((p \wedge \neg q) \to r), \neg r, p\} \vdash q$

Sol.:

1.	$((p \wedge \neg q) \to r)$	PR
2.	$\neg r$	PR
3.	p	PR
4.	$\neg q$	AS
5.	$(p \wedge \neg q)$	$\wedge$I 3,4
6.	r	$\to$E 1,5
7.	$\bot$	$\bot$I 2,6
8.	q	RAA 4-7

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3. Semantics / Argument Validity:

Prove / disprove the following argument is valid:

Premises:

$$\Sigma = \{((\neg p) \to (q \vee r)), ((\neg q) \to ((\neg p) \wedge s)), (s \to (q \vee r))\}$$

Conclusion: q

Sol.: Invalid! I.e., $\Sigma \not\models q$.

To show this, it suffices to find a truth valuation so that $v^*(\Sigma) = T$ but $v^*(q) = F$.

Aside: Coming up with such a solution is hard. In order to do so, you should suppose that such a valuation exists, then determine what properties it might have, like so:

- Since $v^*(q) = F$, need $v^*(\neg q) = T$.
- Then, $\because v^*(\neg q \to (\neg p \wedge s)) = T$, by $\to$ properties, have $v^*(\neg p \wedge s) = T$.
- $\therefore$ By $\wedge$ prop.'s, we have $v^*(\neg p) = T$ (i.e. $v^*(p) = F$) and $v^*(s) = T$.
- Now since $v^*(\neg p \to (q \vee r)) = T$, by $\to$ prop.'s have $v^*(q \vee r) = T$.
- B/c $v^*(q) = F$, by $\vee$ prop.'s, $v^*(r) = T$.

Thus we have our valuation:

$$v: \begin{array}{l} p \mapsto F \\ q \mapsto F \\ r \mapsto T \\ s \mapsto T \end{array}$$

works. □

Alt. Sol.'n (using truth tables...) ↓ ...

Solution 1: We construct the following truth table for all the formulas involved.

p	q	r	s	$(\neg p) \to (q \vee r)$	$(\neg q) \to ((\neg p) \wedge s)$	$s \to (q \vee r)$	q	
1	1	1	1	1	1	1	1	$*$
1	1	1	0	1	1	1	1	$*$
1	1	0	1	1	1	1	1	$*$
1	1	0	0	1	1	1	1	$*$
1	0	1	1	1	0	1	0	
1	0	1	0	1	0	1	0	
1	0	0	1	1	0	0	0	
1	0	0	0	1	0	1	0	
0	1	1	1	1	1	1	1	$*$
0	1	1	0	1	1	1	1	$*$
0	1	0	1	1	1	1	1	$*$
0	1	0	0	1	1	1	1	$*$
0	0	1	1	1	1	1	0	$\diamond$
0	0	1	0	1	0	1	0	
0	0	0	1	0	1	0	0	
0	0	0	0	0	0	1	0	

The rows that matter for assessing the argument (i.e. the rows on which all the premise formulas are true) are marked with $*$ or $\diamond$. The conclusion formula is false on the last such row (the row marked with $\diamond$). The presence of such a row witnesses that the argument is **not valid**.

□ ↑ (From Wi '25, Tut 02)

~ Fin. ~