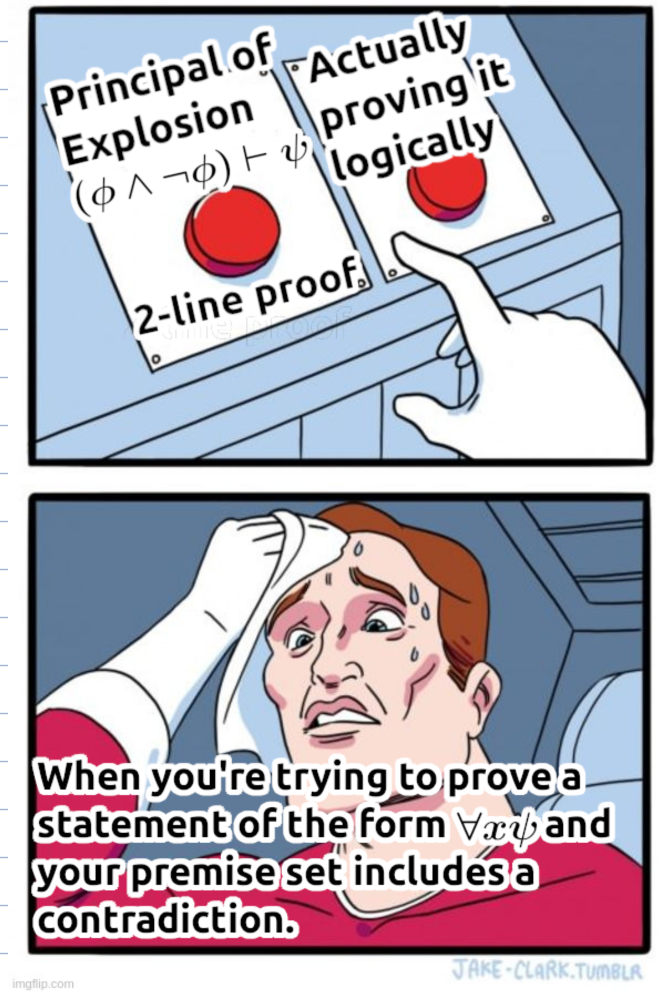
CS 245: TUT 105 - Tutorial 09

Last time: • Semantics

This time: • FOL Formal Proofs
• Examples.

Meme(s) of the week:



Review:

We can extend the *Natural Deduction* proof system that we had for Propositional Logic to First Order Logic:

- Proofs are finite sequences of formulas.
- We write $\Gamma \vdash \varphi$ when there is a proof all of whose "PR" lines come from and whose last formula is φ.
- The deduction rules from Propositional logic also apply here.

"Recall":

- Substitution: If v is free in $\varphi \in \mathcal{L}(\sigma)$ and t term, $\varphi[v/t] \in \mathcal{L}(\sigma)$ is obtained by subbing in t for v.
- Partial Substitution: If t, u terms, then $\varphi[t/{*}u] \in \mathcal{L}(\sigma)$ is obtained by subbing in some (or all) instances of t in φ by u.

We have some more deduction rules for FOL:

Universal Elimination rule. When $\forall x \phi$ appears in the proof and t is any term that does not contain any variable which become bound when it is substituted into ϕ, we can eliminate the $\forall$ quantifier using substitution:

```
m. ∀x φ
⋮
n. φ[x/t]                ∀E m
```

Universal Introduction rule. When ϕ appears in the proof and contains a free variable v that does not appear in the premises or undischarged assumption, we can introduce the $\forall$ quantifier:

```
m. φ
⋮
n. ∀x φ[v/x]             ∀I m
```

$\forall$

Existential Introduction rule. When ϕ appears in the proof, t is a term in ϕ, and x is a variable that does not appear in ϕ, we can introduce the $\exists$ quantifier:

```
m. φ
⋮
n. ∃x φ[t/*x]            ∃I m
```

Existential Elimination rule. When v is a variable that does not appear in ϕ or in ψ, then we can eliminate an existential quantifier:

```
i. ∃x φ
j. ⎡ φ[x/v]             AS
   | ⋮
k. ⎣ ψ
⋮
n. ψ                    ∃E i, j-k
```

Identity Elimination rule. For any terms t and u and any formula ϕ,

```
k. t=u
l. φ
⋮
n. φ[t/*u]                 =E k, l
```

and

```
k. t=u
l. φ
⋮
n. φ[u/*t]                 =E k, l
```

Identity Introduction rule. At any line in the proof, we can always add an identity from a term t to itself:

```
n. t = t                =I
```

In lieu of examples, let’s jump right in and do some practice problems…

Practice Problems:

(1) De Morgan's laws:

In Lecture 12, you saw a proof of $\forall x \neg \varphi \vdash \neg \exists x \varphi$

The other three implied laws were left as exercises. Let's do one of them here:

$$\neg \forall x \varphi \vdash \exists x \neg \varphi.$$

Sol'n:

1.	$\neg \forall x \varphi$	PR
2.	$\neg \exists x \neg \varphi$	AS
3.	$\forall x \neg\neg \varphi$	AS
4.	$\neg\neg \varphi[x/v]$	$\forall$E 3
5.	$\varphi[x/v]$	Double negation. See L.06 for pf.
6.	$\forall x \varphi$	$\forall$I 5
7.	$\perp$	$\neg$E 1,6
8.	$\neg \forall x \neg\neg \varphi$	$\neg$I 3-7
9.	$\neg\neg \exists x \neg \varphi$	L.12 result.
10.	$\perp$	$\neg$E 2,9
11.	$\exists x \neg \varphi$	RAA 2-10 $\square$

EXERCISE: Prove the remaining two FO DeMorgan's Laws yourself! :)

2. ∀ & ∃ Distributivity:

(a.) Show $\forall x(\varphi \wedge \psi) \vdash \forall x \varphi \wedge \forall x \psi$.

Pf.:

1.	$\forall x(\varphi \wedge \psi)$	PR		
2.	$\varphi \wedge \psi\,[x/v]$	∀E	1,	v not in $\varphi \wedge \psi$
3.	$\varphi[x/v] \wedge \psi[x/v]$	R	2,	$\because \varphi \wedge \psi\,[x/a] = \varphi[x/a] \wedge \psi[x/a]$
4.	$\varphi[x/v]$	∧E	3	
5.	$\psi[x/v]$	∧E	3	
6.	$\forall x \varphi$	∀I	4	
7.	$\forall x \psi$	∀I	5	
8.	$\forall x \varphi \wedge \forall x \psi$	∧I	7	

□

(b.) Show $\exists x(\varphi \vee \psi) \vdash \exists x \varphi \vee \exists x \psi$.

Pf.:

1.	$\exists x(\varphi \vee \psi)$	PR		
2.	$\varphi \vee \psi\,[x/v]$	AS		
3.	$\varphi[x/v] \vee \psi[x/v]$	R	2	$\because \varphi * \psi\,[x/t] = \varphi[x/t] * \psi[x/t]$
4.	$\varphi[x/v]$	AS		
5.	$\exists x \varphi[v/x]$	∃I	4	
6.	$\exists x \varphi$	R	5	$\because \varphi[v/x] = \varphi(x)$
7.	$\exists x \varphi \vee \exists x \psi$	∨I	6	
8.	$\psi[x/v]$	AS		
9.	$\exists x \psi[v/x]$	∃I	8	
10.	$\exists x \psi$	R	9	(Same reason as 6.)
11.	$\exists x \varphi \vee \exists x \psi$	∨I	10	
12.	$\exists x \varphi \vee \exists x \psi$	∨E	3,	4-7, 8-11
13.	$\exists x \varphi \vee \exists x \psi$	∃E	1,	2-12.

□

EXERCISE: Do the two other directions yourself! :)

3. Group Theory: Recall:

$$\Gamma_G : \left\{ \begin{array}{c} \forall x \forall y \forall z\, (x\cdot(y\cdot z)) = ((x\cdot y)\cdot z) \\ \forall x\, ((x\cdot e) = x \wedge (e\cdot x) = x) \\ \forall x \exists y\, ((x\cdot y) = e \wedge (y\cdot x) = e) \end{array} \right\}$$

(a.) Show $\Gamma_G \vdash (e\cdot e) = e$.

Pf.:

1.	$\forall x\, ((x\cdot e) = x \wedge (e\cdot x) = x)$	PR
2.	$(e\cdot e) = e \wedge (e\cdot e) = e$	$\forall$E 1
3.	$(e\cdot e) = e$	$\wedge$E 2

$\square$

(b.) Show $\Gamma_G \vdash \forall x\, ((x\cdot x) = x \to x = e)$.

Ideas:

- The only way to prove this statement is with \forall introduction; work backwards.
- We want to prove an implication, so we should assume a specialized version of the —> premise, and then prove the conclusion, then invoke (—> I).

Pf.:

1.	$(g\cdot g) = g$	AS
2.	$\forall x \exists y\, ((x\cdot y) = e \wedge (y\cdot x) = e)$	PR
3.	$\exists y\, ((g\cdot y) = e \wedge (y\cdot g) = e)$	$\forall$E 2
4.	$g\cdot h = e \wedge h\cdot g = e$	AS
5.	$g\cdot h = e$	$\wedge$E 4
6.	$\forall x \forall y \forall z\, (x\cdot(y\cdot z)) = ((x\cdot y)\cdot z)$	PR
7.	$\forall y \forall z\, (g\cdot(y\cdot z)) = ((g\cdot y)\cdot z)$	$\forall$E 6
8.	$\forall z\, (g\cdot(g\cdot z)) = ((g\cdot g)\cdot z)$	$\forall$E 7
9.	$g\cdot(g\cdot h) = (g\cdot g)\cdot h$	$\forall$E 8
10.	$g\cdot(g\cdot h) = g\cdot h$	=E 1,9
11.	$g\cdot e = e$	=E 5,10
12.	$\forall x\, (x\cdot e) = x \wedge (e\cdot x) = x$	PR
13.	$(g\cdot e) = g \wedge (e\cdot g) = g$	$\forall$E 12
14.	$(g\cdot e) = g$	$\wedge$E 13
15.	$g = e$	=E 14, 11
16.	$g = e$	$\exists$E 3, 4–15
17.	$(g\cdot g) = g \to g = e$	$\to$I 1–12
18.	$\forall x ((x\cdot x) = x \to x = e)$	$\forall$I 17

$\square$

~ Fin. ~