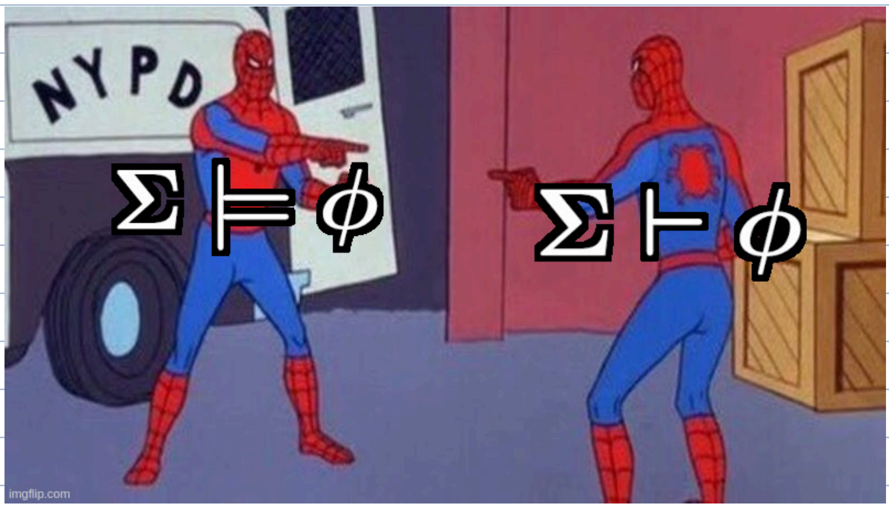
CS 245: TUT 105 - Tutorial 10

Last time:
- FOL Formal Proofs
- Examples.

This time:
- FOL Soundness & Completeness

Meme(s) of the week:



(Again)

In light of recent events

Review:

⚠ Correction to a definition from earlier:

Valuation: To define a valuation $val_{\mathcal{M},s}$, first define a <u>variable assignment</u> $s : \{\text{FOL vars}\} \longrightarrow \mathcal{D}$, where $\mathcal{D}$ is the domain of our structure $\mathcal{M}$.

Then, we can also define:

$$val_{\mathcal{M},s}(x) := s(x), \text{ for any variable } x.$$

The rest of the definition is straightforward, and similar to what we already had.

We note that in the case our formula phi is a sentence, we don't actually need to worry about the variable assignment, and can so just write

$$val_{\mathcal{M}}(\varphi) \text{ instead of } val_{\mathcal{M},s}(\varphi).$$

Also, corrections to errors in formal proof def'ns:

- $\exists E$: The variable v introduced must be new; it can't appear anywhere else in φ, ψ, the premises, or any undischarged assumptions.
- $= E$: Can't always sub t for u or vice versa in φ.

  You can only do this when the variable instances appearing in t or u are not bound in phi.

⚠ In general, the goal is to avoid naming collisions w/ the variables inside φ, as having collisions creates a (semantically) different formula!

— Anyways, onto new stuff... —

Soundness:

Thm.: (Soundness) For any subset of sentences $\Gamma \subseteq L(\sigma)$ & sentence φ, $\Gamma \vdash \varphi \Rightarrow \Gamma \models \varphi$.

Proof strat:

In class, we did something slightly different than what's in the notes.

- We first used our new *expanded* definition of valuation to generalize our definition of logical consequence to formulas, not just sentences:

For any $\Gamma \subseteq L(\sigma)$ and $\varphi \in L(\sigma)$, $\Gamma \models \varphi$ means that for any structure-assignment pair $(\mathcal{M}, s)$ and valuation $val_{\mathcal{M},s} : L(\sigma) \longrightarrow \{T, F\}$,

$$val_{\mathcal{M},s}(\Gamma) = T \Rightarrow val_{\mathcal{M},s}(\varphi) = T.$$

meaning $\forall \psi \in \Gamma$, $val(\psi) = T$

- Then, we proceeded by induction directly on the line number of the proof, showing that each line of a formal proof is a logical consequence of the premises and undischarged assumptions on that line, just like in the Propositional Logic version of Soundness.

Completeness:

Thm.: (Completeness) For any subset of sentences $\Gamma \subseteq L(\sigma)$ & sentence φ, $\Gamma \models \varphi \Rightarrow \Gamma \vdash \varphi$.

Proof strat: Same idea as in Propositional Logic.
- We want to show every consistent set of sentences is satisfiable, i.e. has a model.
- Instead of trying to show this directly, we first show it for a "special class" of consistent sets — (namely, Maximally Consistent sets) — then extend to all consistent sets using Zorn's Lemma.

Def.: A subset $\Gamma \subseteq L(\sigma)$ is:
- consistent if $\Gamma \nvdash \bot$,
- inconsistent if $\Gamma \vdash \bot$,
- maximally consistent if consistent AND $\forall \varphi \in L(\sigma)$, either $\varphi \in \Gamma$ or $\neg\varphi \in \Gamma$.

Prop. 1: $\forall$ maximally consistent $\Gamma \subseteq L(\sigma)$ and sentences φ, ψ:
- $\Gamma \vdash \varphi \Leftrightarrow \varphi \in \Gamma$.
- $\Gamma \nvdash \varphi \Leftrightarrow \varphi \notin \Gamma$.
- $\varphi \wedge \psi \in \Gamma \Leftrightarrow \varphi \in \Gamma$ and $\psi \in \Gamma$
- $\varphi \vee \psi \in \Gamma \Leftrightarrow \varphi \in \Gamma$ or $\psi \in \Gamma$
- $\varphi \to \psi \in \Gamma \Leftrightarrow \varphi \notin \Gamma$ or $\psi \in \Gamma$
- $\varphi \leftrightarrow \psi \in \Gamma \Leftrightarrow$ both or neither of ψ, φ are in Γ.

Pf.: Exercise!

(See Problems Section...)

Besides our useful notion of maximal consistency, we need the set of sentences that we consider to satisfy some additional properties that let us deal with quantifiers.

For this, we introduce another definition:

Def.: $\Gamma \subseteq L(\sigma)$ is called <u>saturated</u> if for any $\varphi \in L(\sigma)$ containing a single free var. x, there is a constant $c \in L(\sigma)$ so that $(\exists x \varphi \to \varphi[x/c]) \in \Gamma$.

If a set is both maximally consistent and saturated, it has the following additional useful properties related to quantifiers:

Prop. 2: Let $\Gamma \subseteq L(\sigma)$ be maximally consistent & saturated. Then, $\forall \varphi \in L(\sigma)$ containing a unique free var., x,

- $\exists x \varphi \in \Gamma \iff$ there exists a closed term t s.t. $\varphi[x/t] \in \Gamma$,
- $\forall x \varphi \in \Gamma \iff$ for any closed term t, $\varphi[x/t] \in \Gamma$.

Pf.: See notes.

Now, we prove the Completness theorem for maximally consistent, saturated sets… actually, a subset of these, namely those consisting of formulas that do not use "=".

GOAL: For any $\Gamma \subseteq L(\sigma)$ which is: (*) $\left\{ \begin{array}{l} \bullet \text{ maximally consistent} \\ \bullet \text{ saturated} \\ \bullet \text{ has no = predicates} \end{array} \right.$

…we want to construct a model M_Γ of Γ so that for any sentence φ, $\mathrm{val}_{M_\Gamma}(\varphi) = T \iff \varphi \in \Gamma$.

Key idea: Build a universe out of terms!

Def.'n: The **term model** M_Γ^* for Γ sat'ing (*) is defined by

- the domain $D_\Gamma := \{\text{terms of } L(\sigma)\}$
- constants $c^{M_\Gamma^*} := c$
- functions $f^{M_\Gamma^*}(t_1, \dots, t_n) := f(t_1, \dots, t_n)$
- n-ary predicates $P^{M_\Gamma^*} := \{(t_1, \dots, t_n) : P(t_1, \dots, t_n) \in \Gamma\}$

Prop. 3: For every closed term t, $\mathrm{val}_{M_\Gamma^*}(t) = t$. Pf: notes.

Using this, we can prove the special case of the Completeness Theorem…

Lemma 4: For any $\Gamma \subseteq L(\sigma)$ sat'ing (*) and $\varphi \in L(\sigma)$ w/o the "=" symbol, $M_\Gamma^* \models \varphi \iff \varphi \in \Gamma$.

Factoring the Term Model:

The M_Γ^* we constructed in the last section doesn't satisfy φ containing $=$. How to fix this? "Mod out" by formulas that result from substituting "$t = t'$".

> **Definition.** Fix any set Γ of formulas. Define the **equivalence relation for** Γ to be the binary relation $\approx_\Gamma$ such that for any two closed terms t and t',
>
> $$t \approx_\Gamma t' \qquad \text{iff} \qquad t = t' \in \Gamma.$$

Proposition 5. For any maximally consistent set Γ of sentences, the relation $\approx_\Gamma$ is relexive, symmetric, and transitive. Furthermore, if $t_1, \ldots, t_n$ are any n closed terms, $i \in \{1, 2, \ldots, n\}$, and t'_i satisfies $t'_i \approx_\Gamma t_i$ then for any n-ary function f,

$$f(t_1, \ldots, t_n) \approx_\Gamma f(t_1, \ldots, t_{i-1}, t'_i, t_{i+1}, \ldots, t_n)$$

and for any n-ary predicate P,

$$P(t_1, \ldots, t_n) \in \Gamma \text{ iff } P(t_1, \ldots, t_{i-1}, t'_i, t_{i+1}, \ldots, t_n) \in \Gamma.$$

PC.: notes.

We can use the relation $\approx_\Gamma$ to partition the set of all closed terms into equivalence classes.

Definition. For any maximally consistent set Γ of formulas, define the **equivalence class** for the closed term t to be

$$[t]_{\approx_\Gamma} = \{t' : t' \approx_\Gamma t\}.$$

We are now ready to introduce the factored term model:

Definition. The **factored term model** for the maximally consistent set Γ of formulas is the structure $M_\Gamma^\approx$ with domain $D_\Gamma^\approx = \{[\mathbf{t}]_{\approx_\Gamma} : t \text{ is a closed term}\}$ where we assign constant c the element

$$c^{M_\Gamma^\approx} = [\mathbf{c}]_{\approx_\Gamma},$$

we define the n-ary function f to take the values

$$f^{M_\Gamma^\approx}([\mathbf{t_1}]_{\approx_\Gamma}, \ldots, [\mathbf{t_n}]_{\approx_\Gamma}) = [\mathbf{f(t_1, \ldots, t_n)}]_{\approx_\Gamma},$$

and we define the n-ary predicate P to be the set

$$P^{M_\Gamma^\approx} = \{([\mathbf{t_1}]_{\approx_\Gamma}, \ldots, [\mathbf{t_n}]_{\approx_\Gamma}) : P(t_1, \ldots, t_n) \in \Gamma\}.$$

When Γ is any maximally consistent set of sentences, the factored term model $M_\Gamma^\approx$ is well-defined and satisfies $\mathrm{val}_{M_\Gamma^\approx}(t) = [\mathbf{t}]_{\approx_\Gamma}$ for every closed term t. We use $M_\Gamma^\approx$ to obtain the completeness theorem for maximally consistent and saturated sets of formulas.

Lemma 6. Every maximally consistent and saturated set Γ of sentences is satisfiable.

PC.: notes.

Finally, we can prove the completeness theorem:

Theorem. *(Completeness Theorem, restated)* For every set Γ of sentences and every sentence ϕ, if $\Gamma \vDash \phi$ then $\Gamma \vdash \phi$.

Proof. As in the proof of the Completeness Theorem for propositional logic in Lecture 8, it suffices to show that whenever Γ is consistent then it is also satisfiable.

Consider any consistent set Γ of sentences over a signature σ. Let σ^+ denote the extension of σ obtained by adding a constant symbol c_{ϕ,x_i} for each pair (ϕ, x_i) where the variable x_i is the only free variable in the formula ϕ. Define $\Gamma^+ = \Gamma \cup \{\phi[x_i/c_{\phi,x_i}] : x_i \text{ is the only free variable in } \phi\}$ to be the extension of Γ over σ^+. The set Γ^+ is saturated by construction. And we can obtain a maximally consistent and saturated set $\Gamma' \supseteq \Gamma^+$ by using the same construction and argument as in the proof of Lemma 4 in Lecture 8.

By Lemma 6, the set Γ' is satisfiable. Let M be any model of Γ'. Since Γ^+ is a subset of Γ', then M is also a model of Γ^+. Define the structure M^- over σ to be the restriction of M over this signature. (In other words, we simply ignore the assignments that M assigns to the c_{ϕ,x_i} constant symbols added to σ^+.) This structure satisfies all the sentences in Γ, so it is also a model of it. Therefore, Γ is satisfiable.

~ End review ~

Practice Problems:

1. Soundness:

Prove Proposition 1.

Sol.: • $\Gamma \vdash \varphi \iff \varphi \in \Gamma$: ($\Leftarrow$): trivial.
($\Rightarrow$): Suppose $\Gamma \vdash \varphi$, AFSOC $\varphi \notin \Gamma$.

Then $\neg\varphi \in \Gamma$, so $\Gamma \vdash \neg\varphi$, so $\Gamma \vdash \varphi \wedge \neg\varphi \vdash \bot$, meaning Γ is inconsistent, a contradiction.

• $\varphi \to \psi \in \Gamma \iff \neg\varphi \in \Gamma$ or $\psi \in \Gamma$:

$$\varphi \to \psi \in \Gamma \underset{\text{1st property}}{\iff} \Gamma \vdash \varphi \to \psi \underset{\text{DNF for } \varphi \to \psi}{\iff} \Gamma \vdash \neg\varphi \vee \psi \underset{\text{1st prop.}}{\iff} \neg\varphi \vee \psi \in \Gamma$$

$$\underset{\text{4th property}}{\iff} \neg\varphi \in \Gamma \text{ or } \psi \in \Gamma.$$

Other parts are similarly easy, but you should prove them in order, that way you can use previous parts to prove later parts.

$\square$

2. Completeness: "Roadmap."

The proof of the Completeness Theorem in first order logic is a difficult proof… it's long, and combines several concepts and ideas that we've seen earlier in the course, plus introduces some new ones.

Whenever you encounter a long and/or difficult proof of a theorem, one helpful exercise in understanding it is to make a "roadmap" stating the result, and outlinining the high-level ingredients to the proof.

One way (my personal favorite) of making such a roadmap is graphically; the roadmap consists of all lemmas, or important "sub-results" or steps in the proof as nodes, and arrows joining them can either be basic logical implication or checking of the hypotheses, or it can be labeled with a high-level description of how the next result in the chain is deduced from the previous one.

These “roadmaps” can be adapted to any level of detail that you like, and can tell you how to systematically decompose a difficult problem into reasonable chunks, and how far along you are in understanding (or creating) a proof. See this post by Terence Tao, which mentions the Blueprint tool for LEAN 4, a similar idea: https://terrytao.wordpress.com/2023/11/18/formalizing-the-proof-of-pfr-in-lean4-using-blueprint-a-short-tour/

Anyways, **EXERCISE:** Create a blueprint for the proof of Completeness.

It should include all the major lemmas / results in the proof, as well as short descriptions of definitions that we introduce in order to get it to work.

(Open-ended. Answers may vary.)

3. Soundness & Completeness:

Let F be any binary relation symbol. Show that **no** Natural Deduction proof exists to witness

$$\{\forall x \exists y F(x,y)\} \vdash \exists y \forall x F(x,y).$$

for any choice of F.

Solution: Let

$$\begin{aligned} A &= \forall x \exists y F(x,y), \text{ and} \\ B &= \exists y \forall x F(x,y). \end{aligned}$$

By the contrapositive of soundness, it suffices to show that the corresponding entailment fails, i.e. it suffices to show that $\{A\} \not\models B$.
Let ν be a valuation with

$$\mathcal{M} \left\{ \begin{aligned} \text{domain, } \mathcal{D} &= \{1, 2\} \\ F^\nu &= \text{equality} \end{aligned} \right.$$

One can show directly that this is a valid counterexample for logical implication:

There are no constant or function symbols. There are no free variables. So ν suffices to interpret all formulas in this question.

(a) **First, we will show that** $A^\nu = 1$.
Consider all possible overrides (u/d) for $d \in \mathcal{D}$.

- $(u/1)$: We have that
$$F(u,w)^{\nu(u/1)(w/1)} = 1$$
because
$$(1,1) \in F^\nu.$$
Then by the satisfaction rule for $\exists$, we have that
$$\exists y\, F(u,y)^{\nu(u/1)} = 1.$$
- $(u/2)$: We have that
$$F(u,w)^{\nu(u/2)(w/2)} = 1$$
because
$$(2,2) \in F^\nu.$$
Then by the satisfaction rule for $\exists$, we have that
$$\exists y\, F(u,y)^{\nu(u/2)} = 1.$$

Therefore, by the satisfaction rule for $\forall$, $A^\nu = 1$.

(b) **Next, we will show that** $B^\nu = 0$.
Consider all possible overrides (w/d) for $d \in \mathcal{D}$.

- $(w/1)$: We have that
$$F(u,w)^{\nu(w/1)(u/2)} = 0$$
because
$$(2,1) \notin F^\nu.$$
Then by the satisfaction rule for $\forall$, $\forall x\, F(x,w)^{\nu(w/1)} = 0$.
- $(w/2)$: We have that
$$F(u,w)^{\nu(w/2)(u/1)} = 0$$
because
$$(1,2) \notin F^\nu.$$
Then by the satisfaction rule for $\forall$, $\forall x\, F(x,w)^{\nu(w/2)} = 0$.

We checked all possible overrides (w/d) and did not find any choice of $d \in \mathcal{D}$ such that $\forall x\, F(x,w)^{\nu(w/d)} = 1$. Thus, by the satisfaction rule for $\exists$, $\exists y\, \forall x\, F(x,w)^\nu = 0$.

Thus, $\mathcal{M} \models A$ but $\mathcal{M} \not\models B$, so $\{A\} \not\models B$, as needed. $\square$ ~fin.~