Coefficient Positivity and Analytic Combinatorics

John Hunn Smith

Supervisors: Rafael Oliveira, Stephen Melczer

Alg & Enum Comb *Pre*-Seminar

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GOAL: Link behavior of F and f_n.

E.g.: Deriving closed form via GF

Fibonacci Numbers: $0, 1, 1, 2, 3, 5, 8, 13, \ldots$

- Satisfy recurrence: $F_n = F_{n-1} + F_{n-2}$ with $F_0 = 0, F_1 = 1$.

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- Satisfy recurrence: $F_n = F_{n-1} + F_{n-2}$ with $F_0 = 0, F_1 = 1$.
- **GOAL:** Derive a closed-form expression for F_n.

$$\begin{aligned} F(x) &= x + \sum_{n=2}^{\infty} F_n x^n \\ &= x + \sum_{n=2}^{\infty} (F_{n-1} + F_{n-2}) x^n \\ &= x + xF(x) + x^2 F(x). \end{aligned}$$

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\implies F(x) &= -\frac{x}{(x+\phi)(x+\tau)} \\
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Start with Cauchy Integral formula:

$$t_n = \frac{1}{2\pi i} \int_{|z|=\epsilon} \tan(z) \frac{dz}{z^{n+1}} = Res_{z=0} \left(\frac{\tan(z)}{z^{n+1}} \right).$$

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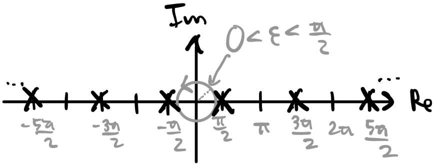

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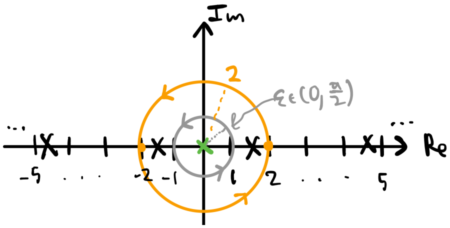

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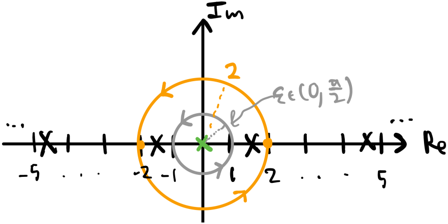

Deriving an asymptotic for t_n:

STEP 1: Introduce an exponentially smaller integral.

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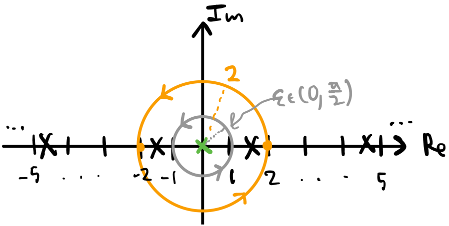

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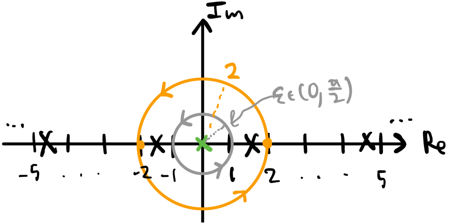

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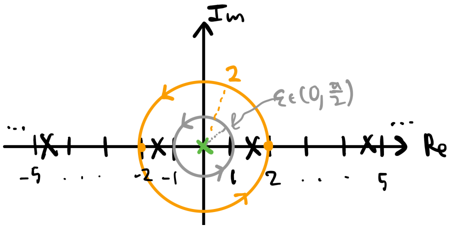

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$$|I_n| \leq len(C) \cdot \max_{|z|=2} \left| \frac{\tan(z)}{(2\pi)z^{n+1}} \right| = 2(2\pi) \frac{M}{(2\pi)2^{n+1}} = O(2^{-n}).$$

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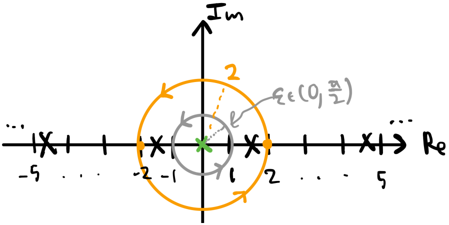

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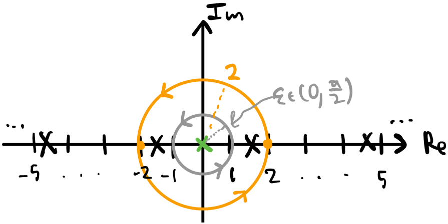

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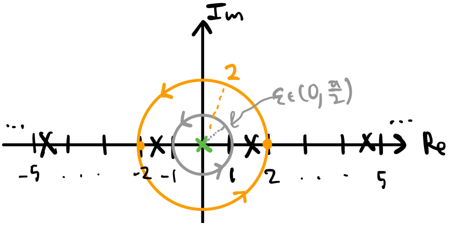

Deriving an asymptotic for t_n:
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$$I_n = Res_{z=0}\left(\frac{\tan(z)}{z^{n+1}}\right) + Res_{z=\pi/2}\left(\frac{\tan(z)}{z^{n+1}}\right) + Res_{z=-\pi/2}\left(\frac{\tan(z)}{z^{n+1}}\right)$$

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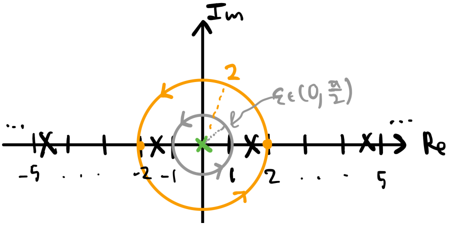

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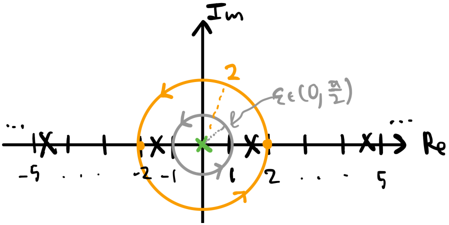

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$$a_n = 2\left(\frac{2}{\pi}\right)^{n+1} n! + O(2^{-n} n!)$$

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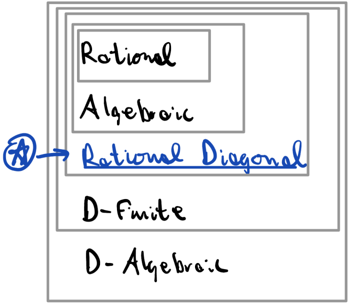

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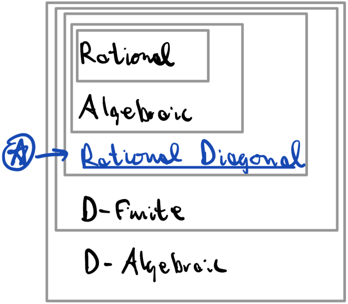

Decidability – Determining Asymptotics:

GF class	Status
Rational	

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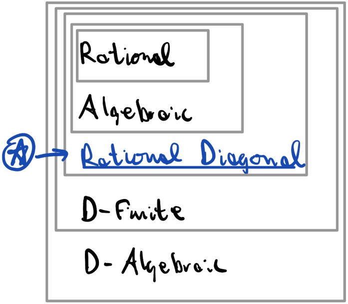

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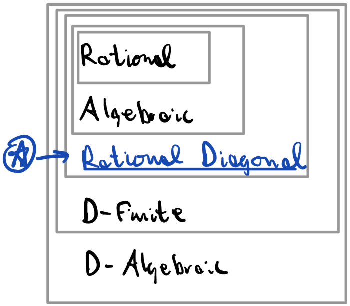

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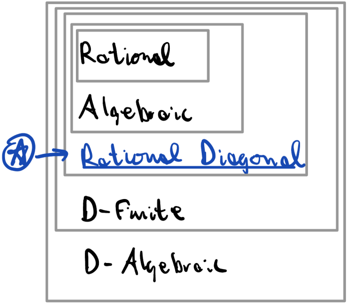

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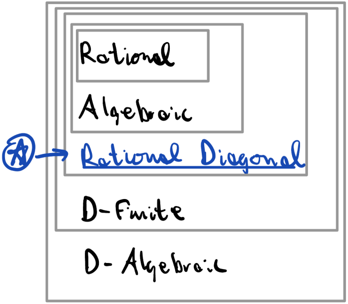

Decidability – Determining Asymptotics:

GF class	Status
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Algebraic	Decidable*
D-Finite	Open
D-Algebraic	Undecidable

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Problem

Complete Positivity Problem (CPP):
Given a C-finite sequence (f_n), decide if $f_n > 0$ for all n.

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Do we think this is easy or hard?

Another question...

When is (f_n) *positive*?

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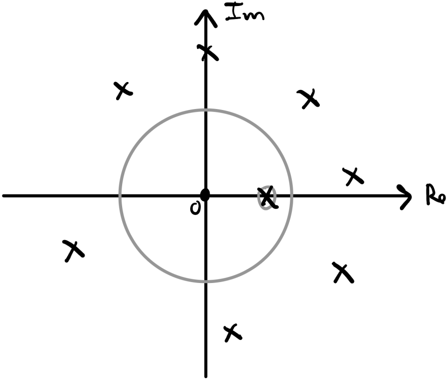
Even deciding positivity for *C-finite* sequences of **order 6** would be groundbreaking!

Nevertheless...

There is a case when we *can* solve CPP.



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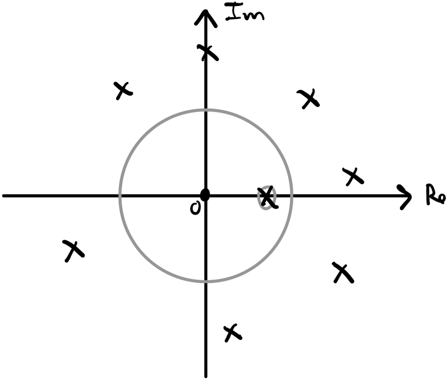

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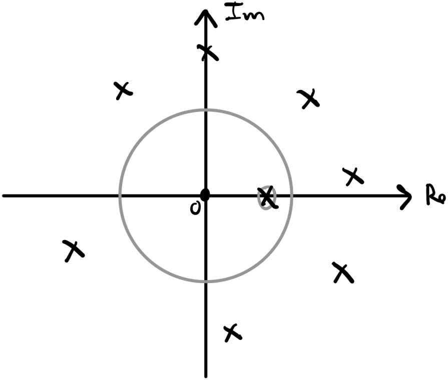

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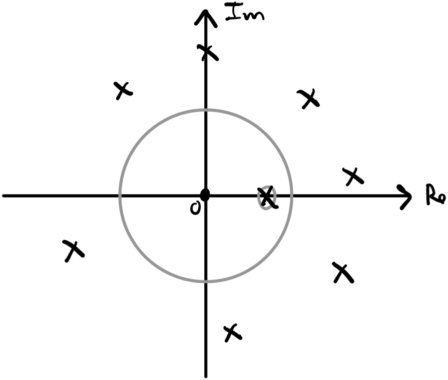

In this case, we can:

1. Determine an asymptotic for f_n
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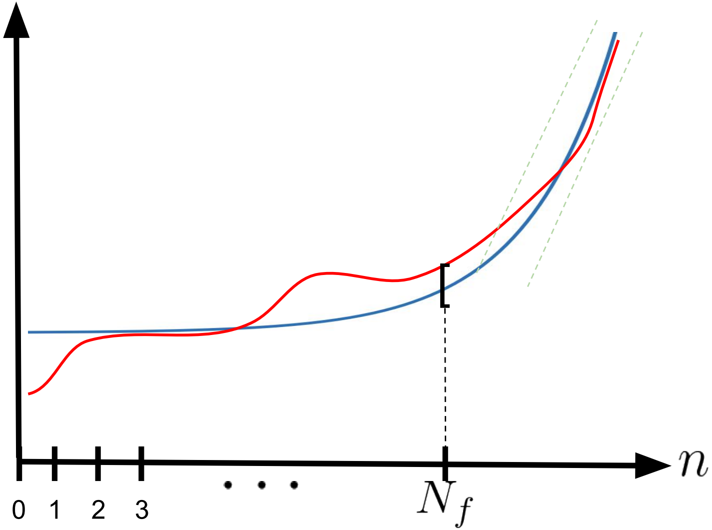
3. Use bound to show that *eventually* f_n is positive.
4. Check the first finitely many terms.

Illustration



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There are (at least) two avenues:

- Larger GF classes...
- Multivariate!

Computing N_f – Others' Work

COMPUTING ERROR BOUNDS FOR ASYMPTOTIC EXPANSIONS OF REGULAR P-RECURSIVE SEQUENCES

RUIWEN DONG, STEPHEN MELCZER, AND MARC MEZZAROBBA

ABSTRACT. Over the last several decades, improvements in the fields of analytic combinatorics and computer algebra have made determining the asymptotic behaviour of sequences satisfying linear recurrence relations with polynomial coefficients largely a matter of routine, under assumptions that hold often in practice. The algorithms involved typically take a sequence, encoded by a recurrence relation and initial terms, and return the leading terms in an asymptotic expansion up to a big-O error term. Less studied, however, are effective techniques giving an explicit bound on asymptotic error terms. Among other things, such explicit bounds typically allow the user to automatically prove sequence positivity (an active area of enumerative and algebraic combinatorics) by exhibiting an index when positive leading asymptotic behaviour dominates any error terms.

In this article, we present a practical algorithm for computing such asymptotic approximations with rigorous error bounds, under the assumption that

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Example

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$$\frac{1}{(1-x)(1-y)+(1-x)(1-z)+(1-y)(1-z)} = \sum_{k,m,n\geq 0} a(k,m,n)x^k y^m z^n.$$

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Is $a(k,m,n)$ positive for each $(k,m,n) \in \mathbb{N}^3$?

$$\frac{1}{(1-x)(1-y)+(1-x)(1-z)+(1-y)(1-z)} = \sum_{k,m,n\geq 0} a(k,m,n)x^k y^m z^n.$$

- First conjectured by H. Lewy and K. Friedrichs in 1930.

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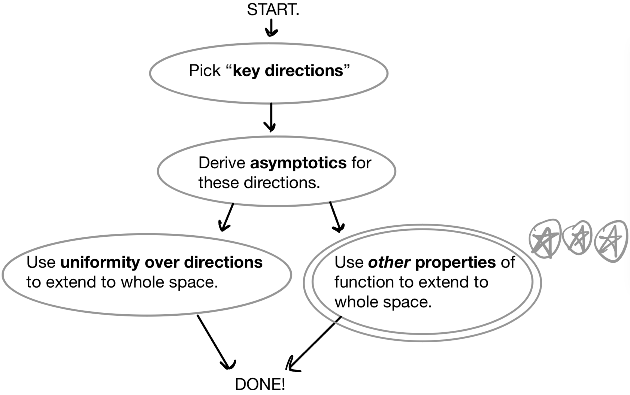
  > *"...the tools used, however, are in no proportion to the simplicity of the statement..." [Sze33]*

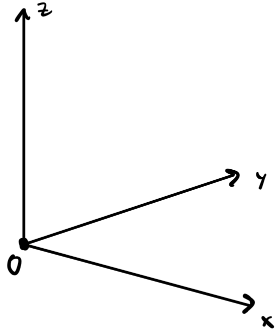
- In 1933 T. Kaluza proved it *again* using only elementary techniques. [Kal33]

Some other classic results

Authors, Year	Function
Asky & Gasper, 1977	$\frac{1}{1-x-y-z+4xyz}$
Scott & Sokal, 2014	$\frac{1}{1-x-y-z-w+\frac{2}{3}(xy+xz+xw+yz+yw+zw)}$
Kauers & Zeilberger, 2007	$\frac{1}{1-x-y-z-w+2(yzw+xzw+xyz+xwy)+4xyzw}$
Gillis, Reznick & Zeilberger, 1983	$\frac{1}{1-x_1-\ldots-x_d+cx_1\cdots x_d}$

Asymptotics-centric approach





ACSV-centric approach

START.

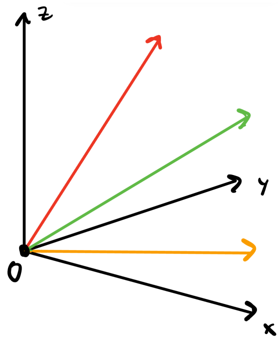
Pick "**key directions**"

Derive **asymptotics** for these directions.

Use **uniformity over directions** to extend to whole space.

Use ***other* properties** of function to extend to whole space.

DONE!



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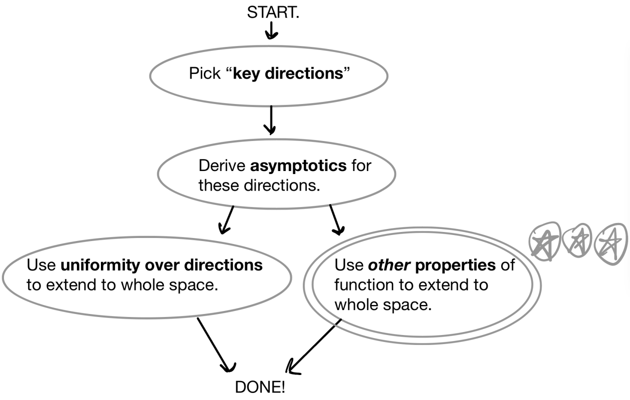
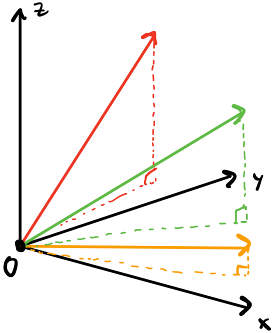




ACSV-centric approach

START.

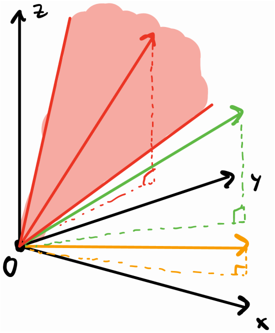
Pick "**key directions**"

Derive **asymptotics** for these directions.

Use **uniformity over directions** to extend to whole space.

Use ***other*** **properties** of function to extend to whole space.

DONE!



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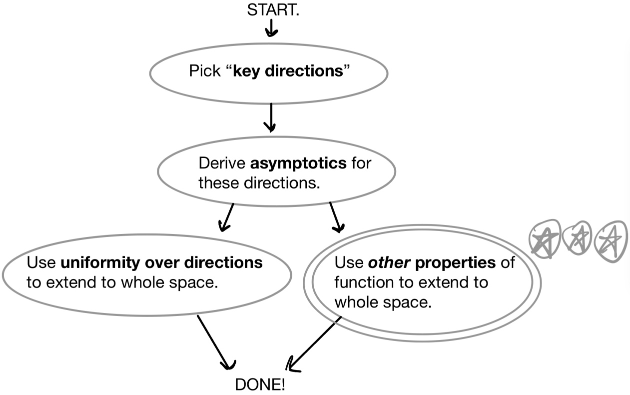

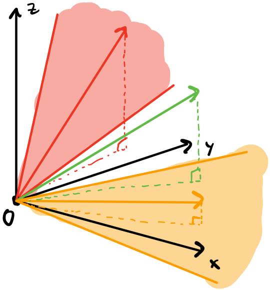

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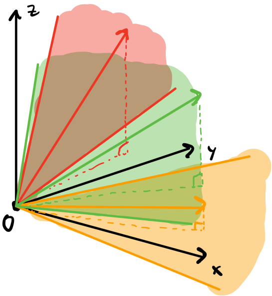
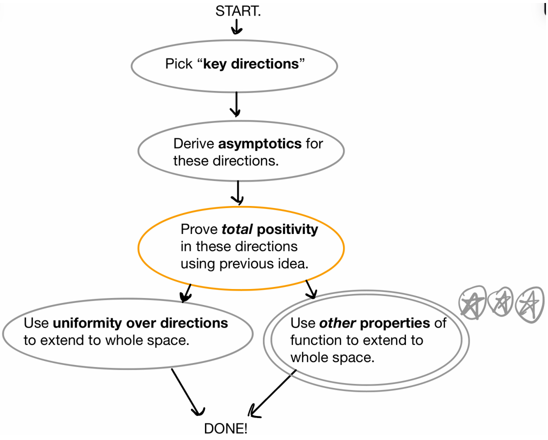


IDEA: Complete positivity from *eventual* positivity



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Suppose we're given a rational function in d complex variables:

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In other words, $f_{n\boldsymbol{r}} > 0$ for n large enough.

IDEA: Complete Positivity from *Eventual* positivity

Then, positivity of *all* coefficients $f_{n\boldsymbol{r}}$ could be determined by:

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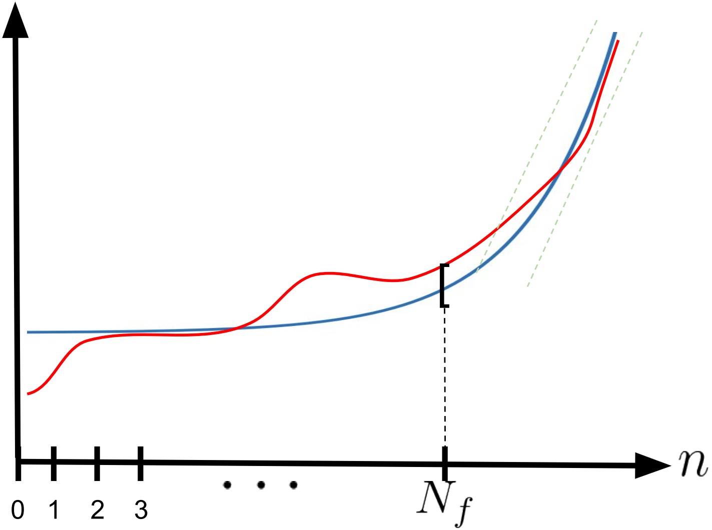
3. Checking positivity for terms up to N_f.

Illustration, again...



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ANSWER: Use techniques from *ACSV*

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The $\boldsymbol{r}$-**diagonal** *of* F *is*

$$\Delta_{\boldsymbol{r}} F(z) = \sum_{n=0}^{\infty} f_{n\boldsymbol{r}} z^n.$$

When $\boldsymbol{r} = \mathbf{1}$ *we simply write* ΔF *for* $\Delta_{\boldsymbol{r}} F$.

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n \ m	0	1	2	3	4	5	6	7	8
0	1	1	1	1	1	1	1	1	1
1	1	3	5	7	9	11	13	15	17
2	1	5	13	25	41	61	85	113	145
3	1	7	25	63	129	231	377	575	833
4	1	9	41	129	321	681	1289	2241	3649
5	1	11	61	231	681	1683	3653	7183	13073
6	1	13	85	377	1289	3653	8989	19825	40081

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By Cauchy Integral formula,

$$f_{\boldsymbol{j}} = \frac{1}{(2\pi i)^d} \int_{T(\boldsymbol{a})} \frac{F(\boldsymbol{z})}{\boldsymbol{z}^{\boldsymbol{j}+\boldsymbol{1}}} d\boldsymbol{z}, \quad j \in \mathbb{N}^d,$$

$T(\boldsymbol{a})$ a small product of circles.

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How ACSV works:

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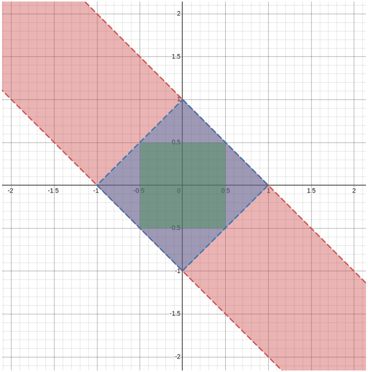
So, $f_{i,j} = \binom{i+j}{j}$.



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First, using a theorem of Smooth ACSV...

Asymptotics of Simple Poles

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*(**Smooth Asymptotics of Simple Poles**):*
Suppose that $F = G/H$ admits a strictly minimal smooth critical point $\boldsymbol{w} \in \mathbb{C}_*^d$ in the direction $\boldsymbol{r} \in \mathbb{R}_*^d$

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where the constants $C_0, \ldots, C_M$ are computable.

What do some of these terms mean?

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(Smooth) critical points *are solutions of*

$$\begin{cases} H(\boldsymbol{z}) = 0, \\ z_1 H_{z_1}(\boldsymbol{z}) - z_j H(\boldsymbol{z})_{z_j} = 0, 2 \leq j \leq d \\ H_{z_j}(\boldsymbol{z}) \neq 0, \ \textit{for some } j. \end{cases}$$

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In our case, the critical point equations become

$$\begin{cases} 1 - x - y = 0, \\ x \cdot (-1) - y \cdot (-1) = 0, \\ H_y = -1 \neq 0, \end{cases}$$

which obviously holds for $\sigma = (1/2, 1/2)$.

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Now, resolve "from scratch."

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Remember the 4 steps...

1. Bound the *exponential growth*, $\rho = \limsup_{n\to\infty} |f_{n\boldsymbol{r}}|^{1/n}$.
2. Determine contributing singularities.
3. "Localize" and compute residues.
4. Apply the Saddle-point method to determine asymptotic.

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Bound the *Exponential Growth*...

- By Cauchy integral formula:

$$\binom{2n}{n} = \frac{1}{(2\pi i)^2} \int_{T(a,b)} \frac{1}{1-x-y} \frac{dxdy}{x^{n+1}y^{n+1}}, \qquad \forall (a,b) \in \mathcal{D}.$$

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- This upper bound is minimized when $(a,b) \to (1/2, 1/2)$.
- This suggests $\binom{2n}{n} \approx 4^n S(n)$, where S grows subexponentially.

E.g.: Central binomial coefficient asymptotics – STEP 2

Determine *contributing* singularities...

- We want singularities with

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- This gives $\sigma := (1/2, 1/2)$. We should "localize" around σ.

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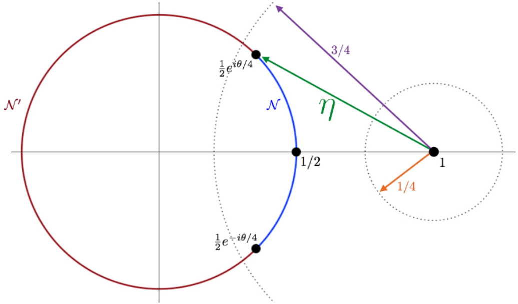

Source: Melczer (2021)

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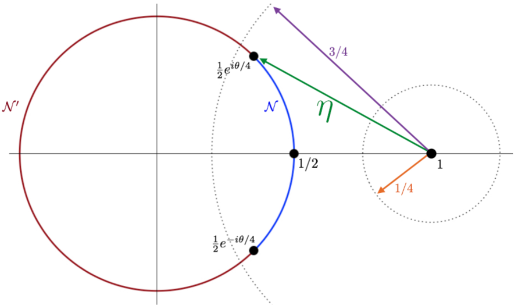

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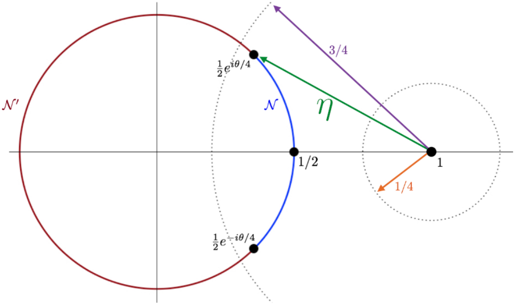

Source: Melczer (2021)

$$I = \binom{2n}{n} = \frac{1}{(2\pi i)^2} \int_{|x|=1/2} \int_{|y|=1/4} \frac{1}{1-x-y} \frac{dy}{y^{n+1}} \frac{dx}{x^{n+1}}.$$

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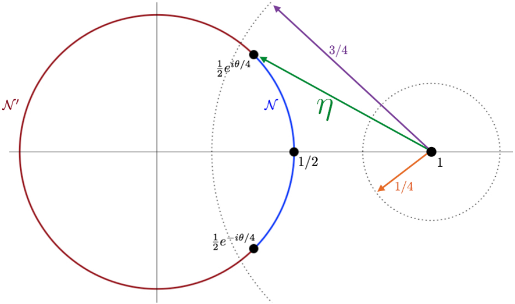

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$$\forall x \in \mathcal{N}, |1-x| < \underbrace{\left|1 - \frac{e^{i\pi/4}}{2}\right|}_{\eta} = 0.7368\ldots \text{ and } \forall x \in \mathcal{N}', |1-x| \geq \eta$$

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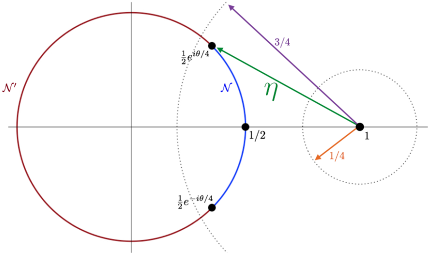

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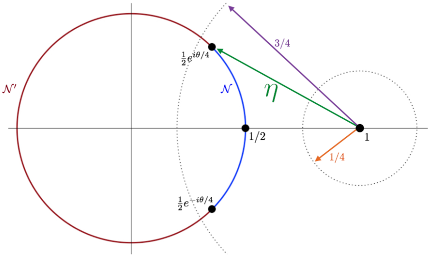

Now consider

$$I_{loc} = \frac{1}{(2\pi i)^2} \int_{\mathcal{N}} \int_{|y|=1/4} \frac{1}{1-x-y} \frac{dy}{y^{n+1}} \frac{dx}{x^{n+1}}.$$

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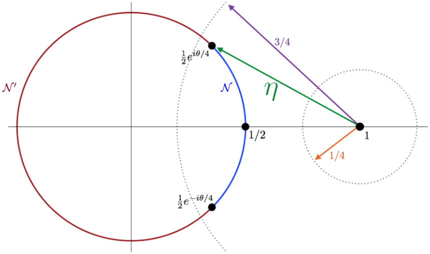

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Can show $|I - I_{loc}|$ exponentially smaller than 4^n.

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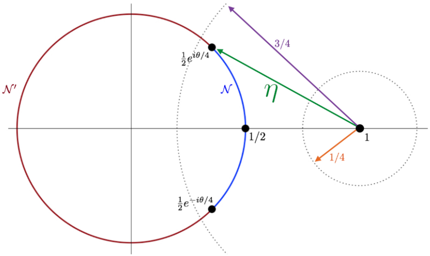

$$I_{out} = \frac{1}{(2\pi i)^2} \int_{\mathcal{N}} \int_{|y|=3/4} \frac{1}{1-x-y} \frac{dy}{y^{n+1}} \frac{dx}{x^{n+1}}.$$

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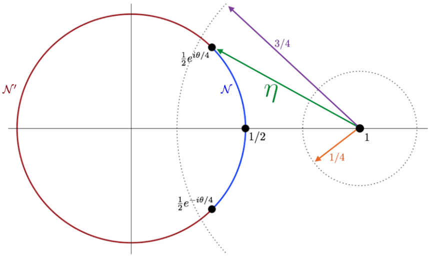

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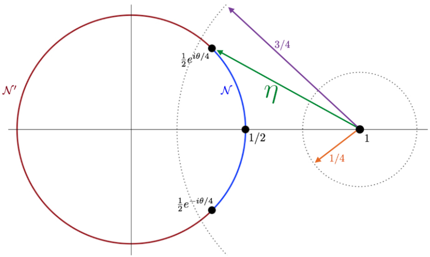

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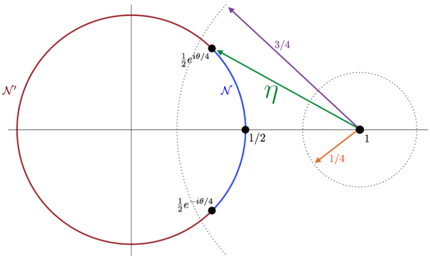

$$\binom{2n}{n} = \underbrace{I_{loc} - I_{out}}_{\chi} + (I - I_{loc}) + I_{out}$$

$$= \frac{1}{(2\pi i)^2} \int_{\mathcal{N}} \left(\int_{|y|=1/4} - \int_{|y|=3/4} \right) + \text{(exp. small)}$$

$$= \frac{1}{2\pi i} \int_{\mathcal{N}} (Res_{y=1-x})\, dx + \text{(exp. small)}$$

$$= \frac{1}{2\pi i} \int_{\mathcal{N}} \frac{dx}{x^{n+1}(1-x)^{n+1}} + \text{(exp. small)}.$$

E.g.: Central binomial coefficient asymptotics – STEP 4

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where

$$A(\theta) = \frac{1}{1 - e^{i\theta}/2} = 2+2i\theta-3\theta^2 \ldots \quad \text{and} \quad \phi(\theta) = \log(2-e^{i\theta})+i\theta = \theta^2+i\theta^3 \ldots$$

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Say $B_n = n^{-2/5}$, so that whenever $|\theta| \geq B_n$ we have $|e^{-n\phi}| \leq e^{-n^{1/5}+O(1)}$.

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$$\int_{-\pi/4}^{\pi/4} Ae^{-n\phi} = \int_{-B_n}^{B_n} + \int_{-\pi/4}^{-B_n} + \int_{B_n}^{\pi/4}$$

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For $|\theta| \leq B_n$ can just take Taylor expansions for A, ϕ and collect error terms:

$$\chi = \frac{4^n}{2\pi}\left(\int_{-B_n}^{B_n} 2e^{-n\theta^2}d\theta\right)\left(1 + O(n^{-1/5})\right).$$

E.g.: Central binomial coefficient asymptotics – STEP 4

Make a change of variables and evaluate on $(-\infty, \infty)$:

$$\int_{-\infty}^{\infty} 2e^{-n\theta^2} d\theta = n^{-1/2} \int_{-\infty}^{\infty} e^{-t^2} dt = 2\sqrt{\pi/n},$$

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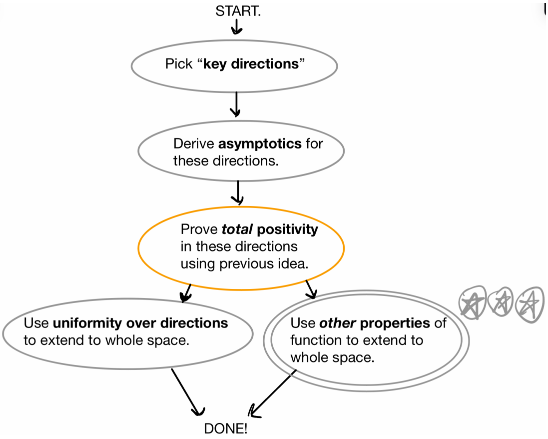
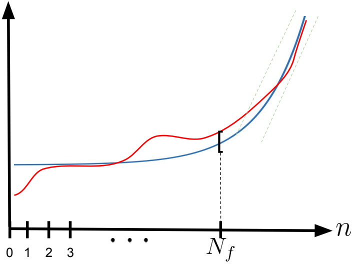
...and add back the tails to get our final expression for χ.

$$\chi = \frac{4^n}{2\pi} \left(\int_{-\infty}^{\infty} 2e^{-n\theta^2} d\theta \right) \left(1 + O(n^{-1/5})\right) = \frac{4^n}{\sqrt{\pi n}} \left(1 + O(n^{-1/5})\right).$$

Almost time for a break...

So, to reiterate:





To be continued...

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