Coefficient Positivity and Analytic Combinatorics

John Hunn Smith

Supervisors: Rafael Oliveira, Stephen Melczer

Alg & Enum Comb *Seminar*

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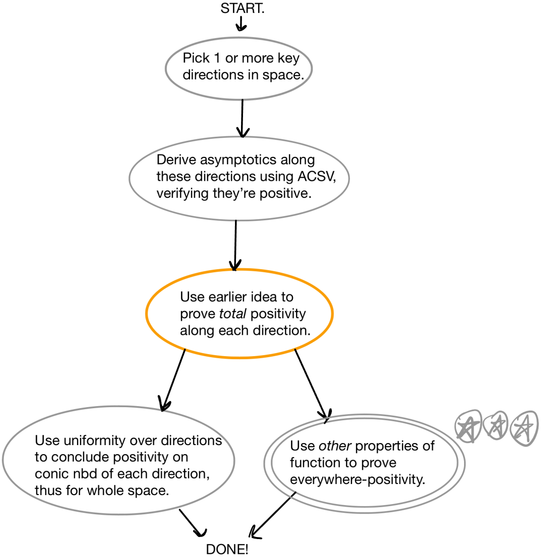
‘Recall’...

Proving total positivity of a sequence *from* eventual positivity:



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Computing N_f – Our Method

INPUT:

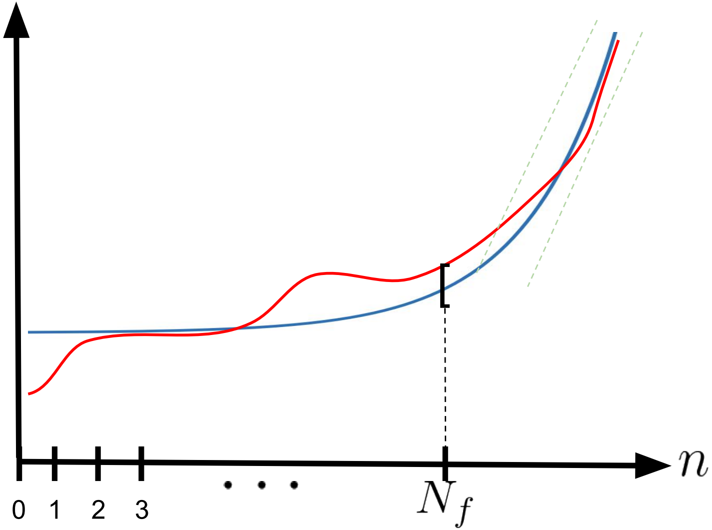
- a rational function $F = G/H$ in d variables
- a direction vector $\boldsymbol{r} \in \mathbb{N}^d$. Here we always take $\boldsymbol{r} = \mathbf{1} := (1, 1, ..., 1)$.

ASSUMING:

- *G, H coprime*
- $H(\mathbf{0}) \neq 0$
- $\mathcal{V} = \mathcal{V}(H)$ *smooth*
- F admits a *nondegenerate strictly minimal smooth critical point* $\boldsymbol{w} \in \mathbb{C}_*^d$.
- The coefficients $f_{n\boldsymbol{r}}$ are eventually positive.

OUTPUT: An $N_f \in \mathbb{N}$ such that $\forall n \geq N_f,\ f_{n\boldsymbol{r}} > 0$.

Computing N_f – Our Method



Computing N_f – Our Method

PROCEDURE:

1. Verify that our set of assumptions hold for our inputs.
2. Run the *smooth ACSV procedure* on our inputs, obtaining:
   - a complex multivariate saddle point integral $\chi(n)$.
   - a function $\lambda(n)$ such that $\chi(n)$ (hence $f_{n\mathbf{r}}$) is asymptotically equivalent to $\lambda(n)$.
3. Compute numbers $\tau \in (0, |\mathbf{w}^{-\mathbf{r}}|)$ and $c > 0$ such that $|f_{n\mathbf{r}} - \chi| < c\tau^n$ for all $n \in \mathbb{Z}_+$.
4. Show that there exists an $L \in \mathbb{Z}_+$ such that $|\chi - \lambda| \leq \lambda - c\tau^n$, for all $n \geq L$.
5. This $L = N_f$. Output it.

DONE.

Computing N_f – Our Method

- Step 3 works because of

Lemma 5.1 *If F admits a strictly minimal smooth contributing point $\mathbf{w} \in \mathbb{C}_*^d$ in the direction $\mathbf{r} \in \mathbb{R}_*^d$ then $|f_{n\mathbf{r}} - \chi| = O(\tau^n)$ for some $\tau < |\mathbf{w}^{-\mathbf{r}}|$.*

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  Lemma 5.1 *If F admits a strictly minimal smooth contributing point* $\mathbf{w} \in \mathbb{C}_*^d$ *in the direction* $\mathbf{r} \in \mathbb{R}_*^d$ *then* $|f_{n\mathbf{r}} - \chi| = O(\tau^n)$ *for some* $\tau < |\mathbf{w}^{-\mathbf{r}}|$.

- Step 4 works because $c\tau^n$ is exponentially smaller than our λ growth rate, and positivity follows from triangle inequality.

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Bivariate Case

We consider functions $F : \mathbb{C}^2 \to \mathbb{C}$ of the form

$$F(x, y) = \frac{1}{1 - ax - by + cxy}, \quad a, b \geq c > 1.$$

- Why? Because it's "easy..."

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$$F(x, y) = \frac{1}{1 - ax - by + cxy}, \quad a, b \geq c > 1.$$

- Why? Because it's "easy..."
- ...But still of practical and academic interest:
    - DeVries, Van der Hoeven, Pemantle (2012)
    - Pemantle & Wilson (2008)
    - Straub & Zudilin (2015)

Bound for family of bivariate functions:

Computation

Let $F : \mathbb{C}^2 \to \mathbb{C}$ be defined by

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Put

$$\boldsymbol{w} = (w_1, w_2) := \frac{ab - \sqrt{(ab)^2 - abc}}{ac}\left(1, \frac{a}{b}\right).$$

To prove positivity of the diagonal coefficients of F it suffices to check the first N_f terms for positivity, where N_f is given by the formula...

Bound for family of bivariate functions:

Computation

$$N_f = \max\left\{N_2, \lceil \delta^{\frac{-1}{\alpha}} \rceil, \left\lceil \left(\frac{c_0}{\mu}\right)^{1/(3\alpha-1)} \right\rceil, \tilde{N}, \left\lceil \left(\frac{c_{16}}{\epsilon}\right)^{1/(3\alpha-1)} \right\rceil, N\right\}$$

where

- $0 < \delta < \min\left\{w_2, \pi/2, \frac{1}{2}\ln(b/(cw_1))\right\}$
- $0 < \epsilon < 1$, $\mu > 0$, *and*
- $\alpha \in \mathbb{Q} \cap (1/3, 1/2)$

are freely chosen parameters, and constants $N_2, c_0, \tilde{N}, c_{16},$ *and* N *are obtained effectively in the derivation.*

E.g.: Bivariate

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Running our procedure above on this F with parameter values

- $\delta = \frac{1}{2} \min(w_2, \pi/2, \log(\sqrt{\frac{b}{cw_1}})) = w_2/2$
- $\epsilon = 1/2$
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yields an index of: $\mathbf{N_f} = \mathbf{1307}$.

Bound for family of bivariate functions:

In the special case when $a = b$, we also have the following corollary:

Corollary

Let F be as above with $a = b > c > 1$. Then, to prove positivity of all of F's power series coefficients, **in any direction**, *it suffices to check the first N_f terms along the diagonal.*

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In the special case when $a = b$, we also have the following corollary:

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Let F be as above with $a = b > c > 1$. Then, to prove positivity of all of F's power series coefficients, **in any direction**, *it suffices to check the first N_f terms along the diagonal.*

Proof: Follows from a result of Straub and Zudilin from 2015.

GRZ Case

Fix an integer $d \geq 4$. We consider the function $F_{d!,d} : \mathbb{C}^d \to \mathbb{C}$ defined by

$$F_{d!,d}(\boldsymbol{z}) = \frac{1}{1 - z_1 \cdots - z_d + d! z_1 \cdot \cdots \cdot z_d}, \quad \forall \boldsymbol{z}.$$

- Originally from Gillis, Reznick and Zeilberger's (1983). The authors considered:

$$F_{c,d}(x_1, ..., x_d) = \frac{1}{1 - x_1 - \cdots - x_d + c x_1 \cdots x_d}.$$

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Conjecture (GRZ)

For any $d \geq 4$, all coefficients of $F_{d!,d}$ are nonnegative.

This has already been shown...

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Positivity of the Rational Function of Gillis, Reznick and Zeilberger

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Abstract

We prove that the power series expansion of the rational function of Gillis, Reznick and Zeilberger (1983) has only nonnegative coefficients.

Reproof using our method

- For $d \geq 4$, GRZ stated that nonnegativity of $F_{d!,d}$ is implied by nonnegativity of $\Delta F_{d!,d}$.
- Thus to prove conjecture, suffices to prove positivity of diagonal!

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- Thus to prove conjecture, suffices to prove positivity of diagonal!

This looks like a job for our method.

Bound for GRZ function:

Computation

Let $d \geq 4$ be an integer. Let $F_{d!,d} : \mathbb{C}^d \to \mathbb{C}$ be defined by

$$F_{d!,d}(\boldsymbol{z}) = \frac{1}{1 - z_1 \cdots - z_d + d! z_1 \cdot \cdots \cdot z_d}, \quad \forall \boldsymbol{z} \in \mathbb{C}^d.$$

To prove positivity of the diagonal coefficients of $F_{d!,d}$ it suffices to check the first N_f terms for positivity, where N_f is given by the formula...

Bound for GRZ function:

Computation

$$N_f = \max\left\{N_2, N_{16}, \tilde{N}, \left\lceil \left(\frac{c_{18}}{\epsilon}\right)^{1/(3\alpha-1)} \right\rceil, N\right\}$$

with the freely chosen parameters:

- $\delta \in (0, \frac{1}{(d-1)^{d-1}})$, *chosen so that* $\frac{1}{\rho\cdot(d-1)} - 1 \geq |e^{2\delta-1}|$, *($\rho$ being the unique real root of a certain polynomial,*
- $0 < \epsilon, \tilde{\epsilon} < 1$
- $\alpha \in (1/3, 1/2) \cap \mathbb{Q}$
- $\mu > 0$,

along with the constants N_{16}, c_{18}, N, $\tilde{N}$ and N_2 which can be computed (mostly) effectively as described in the derivation.

GRZ Example

For $d = 4$ and the following choice of parameters:

- $\epsilon = \tilde{\epsilon} = 1/2$
- $\alpha = 4/10$
- $\mu = 1$
- $\delta = 4/100$

...we end up with

GRZ Example

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...we end up with $\mathbf{N_f = 206271359} \propto \mathbf{10^8}$.

GRZ Example

For $d = 4$ and a *different* choice of parameters:

- $\epsilon = \tilde{\epsilon} = 1/2$
- $\alpha = 3/8$
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...we end up with $\mathbf{N_f = 908295 \propto 10^6}$.

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This may seem prohibitively large, but... stay tuned. :)

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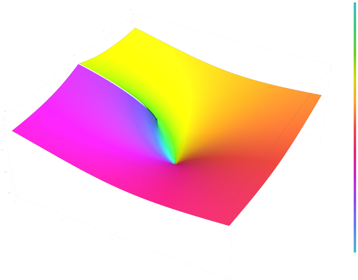
Product Log / Lambert W-function

- *Multivalued function*, with branch $W_k(z)$ for each $k \in \mathbb{Z}$.



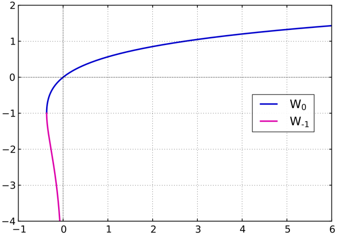


Image credits: Wikipedia.

Product Log / Lambert W-function

- *Multivalued function*, with branch $W_k(z)$ for each $k \in \mathbb{Z}$.
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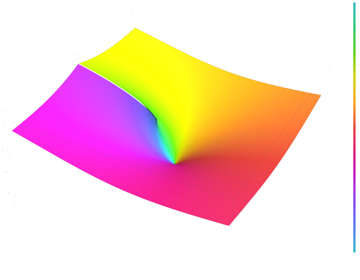

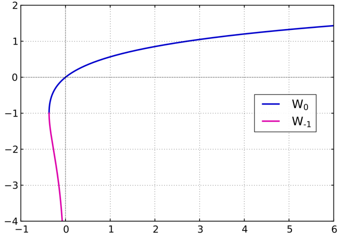

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- For any $z, w \in \mathbb{C}$,
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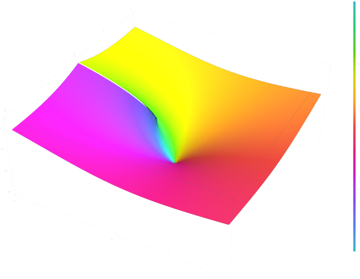
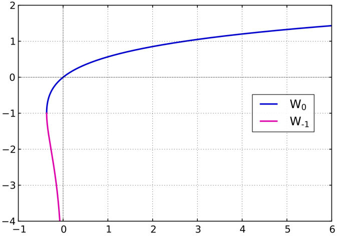
  for some k.
- For real numbers, W_0 and W_{-1} suffice. (See pic at right!)





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"General Use" Lemma 1

Lemma

Calculation involving the Product Log: *Let $\mathcal{H}$ be a positive real, α a rational strictly between 1/3 and 1/2. Then, there exists positive $N \in \mathbb{Z}$ such that*

$$e^{-\frac{\mathcal{H}}{2}n^{1-2\alpha}} < n^{\frac{1}{2}-3\alpha}$$

for all $n \geq N$.

Proof idea:

- We know that $e^{-\frac{\mathcal{H}}{2}n^{1-2\alpha}} < n^{\frac{1}{2}-3\alpha}$ eventually.

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$$\left[\frac{6\alpha-1}{2\alpha-1} W_0\left(\frac{2\alpha-1}{6\alpha-1}\mathcal{H}\right)\right]^{1/(1-2\alpha)}, \left[\frac{6\alpha-1}{2\alpha-1} W_{-1}\left(\frac{2\alpha-1}{6\alpha-1}\mathcal{H}\right)\right]^{1/(1-2\alpha)}$$

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- If none exists, take $N = 1$. $\square$

"General Use" Lemma 2

Lemma

Comparing $c\tau^n$ **and** $\lambda(n)$*:*
Let $c, D > 0$, $\boldsymbol{w} = (w_1, ...w_d) \in \mathbb{C}_*^d$ *and* $0 < \tau < |w_1|^{-1} \cdot ... \cdot |w_d|^{-1}$.
Then, there exists positive $N \in \mathbb{Z}$ *such that*

$$c\tau^n < D|w_1 \cdot ... \cdot w_d|^{-n} n^{(1-d)/2}.$$

for all $n \geq N$.

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- Solve equations for each active branch. Take max to get solution.
- If none are active, just take $N = 1$. $\square$.

Bivariate Bound

Always remember the steps in our Method...

Bivariate Bound

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- F admits a *nondegenerate strictly minimal smooth critical point* $\boldsymbol{w} \in \mathbb{C}_*^d$.
- The coefficients $f_{n\boldsymbol{r}}$ are eventually positive.

OUTPUT: An $N_f \in \mathbb{N}$ such that $\forall n \geq N_f$, $f_{n\boldsymbol{r}} > 0$.

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4. Show that there exists an $L \in \mathbb{Z}_+$ such that $|\chi - \lambda| \leq \lambda - c\tau^n$, for all $n \geq L$.
5. This $L = N_f$. Output it.

DONE.

Bivariate Bound – STEP 1: Verify Assumptions

$$F(x,y) = \frac{1}{1 - ax - by + cxy}, \quad a, b \geq c > 1.$$

- $\mathcal{V}$ is set of zeros of denominator:

$$\mathcal{V} = \left\{ (x,y) \in \mathbb{C}^2 : x \neq \frac{b}{c} \text{ and } y = \frac{ax-1}{cx-b} \right\}.$$

- $\mathcal{V}$ is *smooth*, as $H_x = -a + cy$, $H_y = -b + cx$, so if $(x,y) \in \mathcal{V}$ then $H_y \neq 0$.
- Determine $\mathrm{crit}(\mathbf{1})$ by solving

$$\begin{cases} H(\boldsymbol{w}) = 0 \\ w_1 H_{z_1}(\boldsymbol{w}) - w_j H_{z_j}(\boldsymbol{w}) = 0, \quad 2 \leq j \leq d \end{cases}.$$

Bivariate Bound – STEP 1: Verify Assumptions

- See that

$$\boldsymbol{w} = (w_1, w_2) := \frac{ab - \sqrt{(ab)^2 - abc}}{ac}\left(1, \frac{a}{b}\right) \in \mathrm{crit}(\mathbf{1})$$

  is *minimal* by showing that for no $\boldsymbol{v} \in \mathcal{V}$ is $|\boldsymbol{v}| = t|\boldsymbol{w}|$, for some $t \in (0, 1)$.
- Minimality of $\boldsymbol{w}$ is also *strict*.
- Nondegeneracy of $\boldsymbol{w}$ follows from $\mathcal{H}$ matrix formula.

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By Theorem 5.2 of [Mel21] we obtain

$$f_{n\mathbf{1}} = \frac{(w_1 w_2)^{-n}}{\sqrt{2\pi n}} \cdot \frac{1}{\sqrt{\det(\mathcal{H})}w_2(b - cw_1)}(1 + O(1/n)),$$

where $\mathcal{H}$ is the matrix obtained in Lemma 5.5.

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5. This $L = N_f$. Output it.

DONE.

Bivariate Bound – STEP 3: Finding τ, c

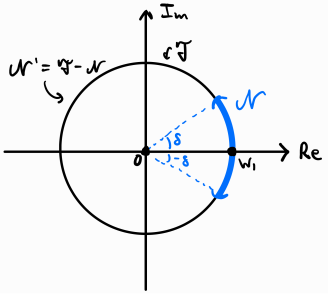
- Call $\lambda(n) := \frac{(w_1 w_2)^{-n}}{\sqrt{2\pi n}} \cdot \frac{1}{\sqrt{\det(\mathcal{H})} w_2 (b - c w_1)}$.

Bivariate Bound – STEP 3: Finding τ, c

- Call $\lambda(n) := \frac{(w_1 w_2)^{-n}}{\sqrt{2\pi n}} \cdot \frac{1}{\sqrt{\det(\mathcal{H})} w_2 (b - c w_1)}$.
- To find χ, we let

$$\delta < \min\left\{ w_2, \pi/2, \frac{1}{2}\ln(b/(cw_1)) \right\}$$

and then introduce $\mathcal{T} = T(\boldsymbol{w}_1)$, $\mathcal{N}, \mathcal{N}'$:



Bivariate Bound – STEP 3: Finding τ, c

- Call $\lambda(n) := \frac{(w_1 w_2)^{-n}}{\sqrt{2\pi n}} \cdot \frac{1}{\sqrt{\det(\mathcal{H})} w_2 (b - c w_1)}$.
- To find χ, we let

$$\delta < \min\left\{ w_2, \pi/2, \frac{1}{2}\ln(b/(c w_1)) \right\}$$

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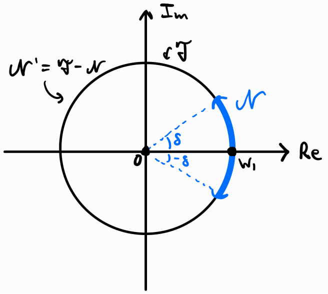

Also define
$g : \mathcal{N} \to \mathbb{C}$, $g(x) := \frac{ax-1}{cx-b}$.

Bivariate Bound – STEP 3: Finding τ, c

- Then, we need to verify *explicitly* that this $\delta, \mathcal{T}, \mathcal{N}, g$ allow us to perform a local singularity analysis.

Bivariate Bound – STEP 3: Finding τ, c

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- Specifically, we need $\eta, \zeta > 0$ such that
  - $|g(\hat{\boldsymbol{z}})| \leq \eta < w_2 + \delta$ for $\hat{\boldsymbol{z}} \in \mathcal{N}$,
  - $w_2 < \zeta \leq |g(\hat{\boldsymbol{z}})|$ for $\hat{\boldsymbol{z}} \in \mathcal{N}'$.

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Bivariate Bound – STEP 3: Finding τ, c

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- We find that $\eta, \zeta := h(\delta)$ works.

Bivariate Bound – STEP 3: Finding τ, c

The rest of this is in a similar vein to the example...

- Introduce the integrals $I, I_{loc}, I_{out}, \chi := I_{loc} - I_{out}$ to compute residues.

Bivariate Bound – STEP 3: Finding τ, c

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- Introduce the integrals $I, I_{loc}, I_{out}, \chi := I_{loc} - I_{out}$ to compute residues.
- Show (explicitly) that $|I - I_{loc}|$ and $|I_{out}|$ grow exponentially smaller than λ.

Bivariate Bound – STEP 3: Finding τ, c

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- Introduce the integrals $I, I_{loc}, I_{out}, \chi := I_{loc} - I_{out}$ to compute residues.
- Show (explicitly) that $|I - I_{loc}|$ and $|I_{out}|$ grow exponentially smaller than λ.
- Combine bounds to obtain c, τ such that

$$|f_{n\mathbf{1}} - \chi| < c\tau^n.$$

Bivariate Bound

PROCEDURE:

1. Verify that our set of assumptions hold for our inputs.
2. Run the *smooth ACSV procedure* on our inputs, obtaining:
   - a complex multivariate saddle point integral $\chi(n)$.
   - a function $\lambda(n)$ such that $\chi(n)$ (hence $f_{n\boldsymbol{r}}$) is asymptotically equivalent to $\lambda(n)$.
3. Compute numbers $\tau \in (0, |\boldsymbol{w}^{-\boldsymbol{r}}|)$ and $c > 0$ such that $|f_{n\boldsymbol{r}} - \chi| < c\tau^n$ for all $n \in \mathbb{Z}_+$.
4. Show that there exists an $L \in \mathbb{Z}_+$ such that $|\chi - \lambda| \leq \lambda - c\tau^n$, for all $n \geq L$.
5. This $L = N_f$. Output it.

DONE.

Bivariate Bound – STEP 4: Finding N_f

Basically the same analysis as the example...

Bivariate Bound – STEP 4: Finding N_f

Basically the same analysis as the example...
...except that we replace all $O(\cdot)$'s with *explicit error bounds.*

Bivariate Bound – STEP 4: Finding N_f

Main steps in our analysis:

- Parametrize $\mathcal{N}$. Write residue χ as a parametrized integral, identifying $A(\theta), \phi(\theta)$:

$$\chi = \frac{(w_1 w_2)^{-n}}{2\pi} \int_{-\delta}^{\delta} A(\theta) e^{-n\phi(\theta)} d\theta$$

Bivariate Bound – STEP 4: Finding N_f

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- Split up the χ-integral like so:

$$\int_{-\delta}^{\delta} = \int_{-\delta}^{-B_n} + \int_{-B_n}^{-B_n} + \int_{B_n}^{\delta}.$$

Bivariate Bound – STEP 4: Finding N_f

$$\int_{-\delta}^{\delta} = \int_{-\delta}^{-B_n} + \int_{-B_n}^{-B_n} + \int_{B_n}^{\delta}.$$

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Bivariate Bound – STEP 4: Finding N_f

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Bivariate Bound – STEP 4: Finding N_f

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- Apply Lemma 1.

Bivariate Bound – STEP 4: Finding N_f

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- Estimate the $[B_n, B_n]$ by a standard Gaussian integral.
- Then "add back the tails."
- Apply Lemma 1.
- Compile error bounds in a table, propagate forward.

Bivariate Bound – STEP 4: Finding N_f

j	Constant $\mathbf{N_j}$	Constant $\mathbf{c_j}$
0	$z_0 = 4\delta/3$	$\frac{8^3}{81}\left[\max\left\{ln\left(\frac{aw_1e^{3\delta/2}+1}{\lvert cw1e^{-3\delta/2}-b\rvert}\right),0\right\}+\max\{ln\lvert 1/w_2\rvert,0\}+\pi+3\delta/2\right]$
1	$z_0 = 4\delta/3$	$8\cdot\frac{1}{1-aw_1e^{3\delta/2}}$
2	$\lceil\delta^{-1/\alpha}\rceil$	c_0
3	N_2	$2\delta e^{c_2}(\frac{1}{1-aw_1}+c_1\delta)$
4	N_3	c_1
5	N_2	c_0
6	$\max(N_2,\lceil\left(\frac{c0}{\mu}\right)^{1/(3\alpha-1)}\rceil)$	$e^{\mu}c_5$
7	N_4	$(1-aw_1)c_4$
8	$\max\{N_6,N_4\}$	$(1-aw_1)c_6c_4$
9	$\max\{N_6,N_7,N_8,1\}$	$\frac{1}{2}\sqrt{\frac{2\pi}{\mathcal{H}}}$
10	N_9	$2c_9$
11	N_9	c_9c_{10}
12	$\max\{N_{11},\tilde{N}\}$	$c_{11}c_{10}^{-1}$
13	$\max\{N_{10},\tilde{N}\}$	1
14	$\max\{N_9,N_{12},N_{13}\}$	$c_9+c_{12}+c_{13}$
15	$\max\{N_3,\tilde{N}\}$	$(1-aw_1)c_3\sqrt{\frac{2\pi}{\mathcal{H}}}$
16	$\max\{N_{14},N_{15}\}$	$c_{14}+c_{15}$

Bivariate Bound – STEP 4: Finding N_f

Bivariate Bound – STEP 4: Finding N_f

- Finally, apply Lemma 2 to find N such that $\epsilon\lambda \leq \lambda - c\tau^n$ whenever $n \geq N$.
- $N_f = \max\{N_{16}, \left\lceil \left(\frac{c_{16}}{\epsilon}\right)^{1/(3\alpha-1)} \right\rceil, N\}$

Bivariate Bound – STEP 4: Finding N_f

- Finally, apply Lemma 2 to find N such that $\epsilon\lambda \leq \lambda - c\tau^n$ whenever $n \geq N$.
- $N_f = \max\{N_{16}, \left\lceil \left(\frac{c_{16}}{\epsilon}\right)^{1/(3\alpha-1)} \right\rceil, N\}$
- By above estimate,
$$|\chi - \lambda| \leq \lambda - c\tau^n$$
for all $n \geq N_f$

...And we're done!

GRZ Bound Proof Summary

Starting with $d \geq 4$ integer,

$$F(\boldsymbol{z}) = F_{d!,d}(\boldsymbol{z}) = \frac{1}{H(\boldsymbol{z})}, \qquad H(\boldsymbol{z}) = 1 - \sum_{i=1}^{d} z_i + d! \prod_{i=1}^{d} z_i.$$

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Baryshnikov, Melczer, Pemantle and Straub showed that:

- $\forall d \geq 4$, $P(x) = 1 - dx + d!x^d$ has a unique root $\rho \in [\frac{1}{d}, \frac{1}{d-1})$
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- $\boldsymbol{\rho} := (\rho, \ldots, \rho)$ is strictly minimal smooth critical.
- Nondegeneracy follows from standard determinant identities.

GRZ Bound Proof Summary

Step 3 (and also Step 2) are both standard, until...

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$$\min_{\hat{\boldsymbol{z}} \in \mathcal{N}'} |g(\hat{\boldsymbol{z}})| \quad \text{where} \quad g(\hat{\boldsymbol{z}}) := \frac{1 - z_1 - \cdots - z_{d-1}}{1 - d! z_1 \ldots z_{d-1}}$$

and $\mathcal{N}' = T(\hat{\boldsymbol{\rho}}) - \mathcal{N}$.

GRZ Bound Proof Summary

Conjecture

$$\rho + \delta > \min_{\hat{\boldsymbol{z}} \in \mathcal{N}'} |g(\hat{\boldsymbol{z}})| > \rho = |g(0, 0, \ldots, 0)|,$$

with $\zeta = \min_{\hat{\boldsymbol{z}} \in \mathcal{N}'} |g(\hat{\boldsymbol{z}})|$ *being computable to arbitrary accuracy.*

GRZ Bound Proof Summary

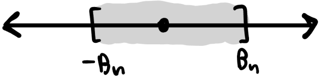
Step 4 is similar, with some wrinkles:

- When computing Taylor series expansions, such as

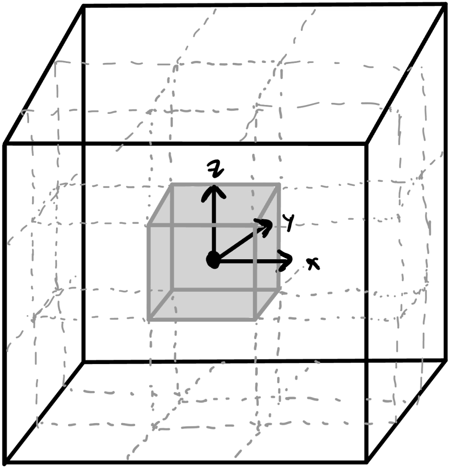
$$\phi(\hat{\boldsymbol{\theta}}) = \frac{1}{2}\hat{\boldsymbol{\theta}}^\intercal \mathcal{H}\hat{\boldsymbol{\theta}} + O(||\hat{\boldsymbol{\theta}}||_1^3), \quad ||\hat{\boldsymbol{\theta}}||_1 \to 0$$

  need to compute upper bounds for images of compact set under derivatives of ϕ.
- Instead of splitting up domain of integration into 3 parts, we split space into $(d-1)^{d-1}$ subregions.

GRZ Bound Proof Summary



vs.



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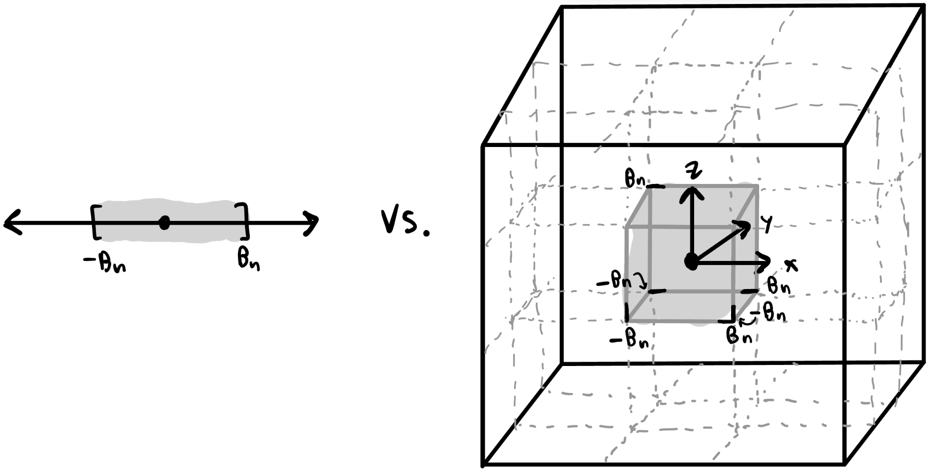


GRZ Bound Proof Summary

- When estimating "central" region integral

$$\int_{[-B_n,B_n]^{d-1}} e^{-\frac{n}{2}\hat{\boldsymbol{\theta}}^\intercal \mathcal{H}\hat{\boldsymbol{\theta}}} d\hat{\boldsymbol{\theta}},$$

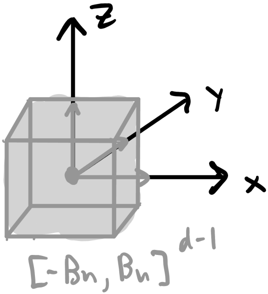
  need to eliminate cross-terms in quadratic form $\hat{\boldsymbol{\theta}}^\intercal \mathcal{H}\hat{\boldsymbol{\theta}}$.
- To do this, use *Principal Axis Theorem*, which says $\exists$ orthogonal Q s.t.

$$Q^\intercal \mathcal{H} Q = \begin{bmatrix} \lambda_1 & 0\dots & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & \dots & 0 \\ \vdots & & \ddots & & \\ 0 & \dots & 0 & 0 & \lambda_{d-1} \end{bmatrix}.$$

- Thus we get $\boldsymbol{y} := Q^{-1}\boldsymbol{\theta}$ s.t. $\boldsymbol{y}^\intercal \mathcal{H}\boldsymbol{y} = \lambda_1 y_1^2 + \cdots + \lambda_{d-1} y_{d-1}^2$.

GRZ Bound Proof Summary

Domain:



Integral:

$$I = \int_{[-B_n, B_n]^{d-1}} e^{-\frac{n}{2}\hat{\boldsymbol{\theta}}^\intercal \mathcal{H} \hat{\boldsymbol{\theta}}} d\hat{\boldsymbol{\theta}}$$

GRZ Bound Proof Summary

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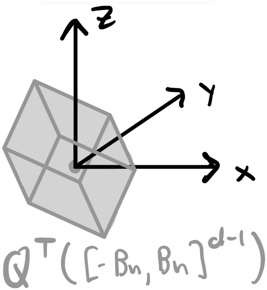

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$$I = \int_{Q^{\intercal}([-B_n,B_n]^{d-1})} e^{-\frac{n}{2}(\lambda_1 y_1^2 + \cdots + \lambda_{d-1} y_{d-1}^2)} d\boldsymbol{y}$$

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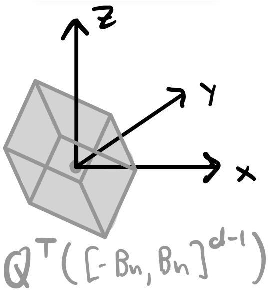

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...But we don't actually want to evaluate this! We want to evaluate on $\mathbb{R}^{d-1}$ and "add back the tails!"

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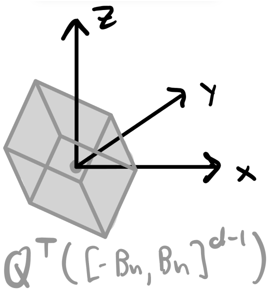

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So, what do we do?

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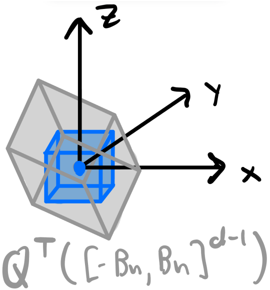

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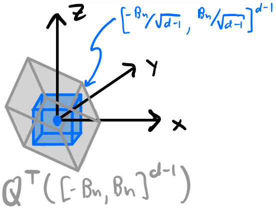

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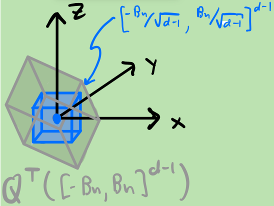
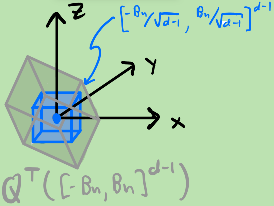


Integral:

$$J := \int_{\mathbb{R}^{d-1} - (\text{smaller cuboid})} A_n(\boldsymbol{y}) e^{-n\phi_n(\boldsymbol{y})} d\boldsymbol{y}$$

GRZ Bound Proof Summary

Domain:



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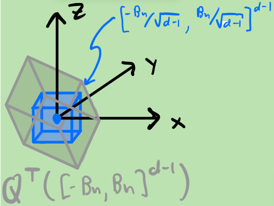
Can show that

$$J \leq \frac{(d-1)^{d-1}-1}{2} e^{-\frac{\lambda_1}{2(d-1)} n^{1-2\alpha}}.$$

GRZ Bound Proof Summary

Domain:



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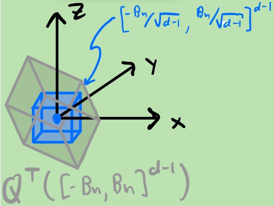

Since

$$I = \int_{\mathbb{R}^{d-1}} - \int_{\mathbb{R}^{d-1} - Q^\intercal([-B_n, B_n]^{d-1})}$$

GRZ Bound Proof Summary

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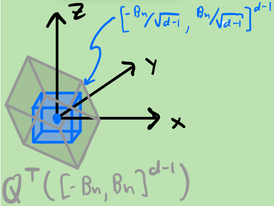

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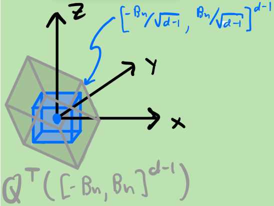

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GRZ Bound Proof Summary

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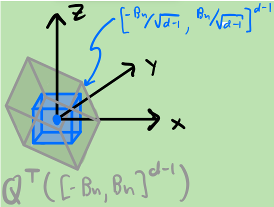

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Can sub in bound for J to our expression for I. The rest of it is the same.

GRZ Bound Proof Summary

Index i	**Value** N_i	**Value** c_i
1	$z_1 := 3\delta/2$	$(d-1)^{d-1}M$
2	$\lceil 1/\delta \rceil$	$e^{-\delta}$
3	N_2	$\frac{(2\delta)^{d-1}e^{c_2}}{1-(d-1)\rho e^{\delta}}$
4	N_3	$((d-1)^{d-1}-1)c_3$
5	$\left\lceil \left(\frac{1}{\delta}\right)^{1/\alpha} \right\rceil$	$(d-1)\tilde{M}$
6	N_5	$(d-1)^{d-1}M$
7	$\max(N_5, \lceil \tilde{\epsilon}^{1/(1-3\alpha)} \rceil)$	$e^{\tilde{\epsilon}}c_6$
8	N_7	$(1-(d-1)\rho)c_7$
9	$\max(N_5, N_7)$	c_5c_7
10	N_9	$(1-(d-1)\rho)c_9$
11	$\max(N_7, N_8, N_{10})$	$c_7 + c_8 + c_{10}$
12	3	$\frac{(d-1)^{d-1}}{2}$
13	N_12	$\frac{c_{12}}{1-(d-1)\rho}$
14	$\max(N_{12}, N_{11})$	$c_{13}c_{11}$
15	$\max(N_{13}, N_{14})$	$c_{13} + c_{14}$
16	$\max(N_{15}, N_4)$	$c_{15} + c_4$
17	$\max(N_{16}, \tilde{N})$	c_{16}
18	$\max(N_{17}, N_{11})$	$c_{17} + c_{11}$

Table of Contents

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ANSWER:

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Is it practical to check the first N_f terms on a computer?

ANSWER:

Sometimes...

E.g.: Bivariate

Take $a = 3, b = 4, c = 3$.

E.g.: Bivariate

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Alternatively, using diagonal recurrence, takes a few milliseconds. :)

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  - Identity involving t then follows by summing over k.

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- Use recurrence to compute values.

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For $d = 4$ and the following choice of parameters:

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In general, cannot expect to be able to compute N_f diagonal coefficients easily.

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Thank you!

Thank you! :)

Stephen Melczer.
The Theory of ACSV for Smooth Points, pages 185–246.
Springer Nature Switzerland, Cham, 2021.