Computing Asymptotic-Guaranteeing Index for Multivariate Generating Functions using ACSV

John H. Smith

Advisor: Yuliy Baryshnikov

ECE 496/499 Senior Project

Table of Contents

The Problem...

Given a function $F : \mathbb{C}^d \to \mathbb{C}$ which is *analytic* at the origin, can one decide whether all its power series coefficients are positive?

The Problem...

Given a function $F : \mathbb{C}^d \to \mathbb{C}$ which is *analytic* at the origin, can one decide whether all its power series coefficients are positive?
E.g.

$$\frac{1}{(1-x)(1-y)+(1-x)(1-z)+(1-y)(1-z)} = \sum_{k,m,n\geq 0} a(k,m,n)x^k y^m z^n$$

My approach: “Classical” positivity vs. ACSV

“Classical” Positivity Proofs:

- Somewhat ad-hoc.
- One function / family of functions per paper.
- Every family is different! So, a variety of techniques have been employed...

ACSV “Eventual Positivity” Proofs:

- More systematic / general.
- Potential to be made effective! (At least for certain classes of functions.)
- Still in development...

Table of Contents

What is analytic combinatorics?

It's in the name...

What is analytic combinatorics?

It's in the name...

$$\text{Combinatorics} + \text{Analysis}$$

What is analytic combinatorics?

It's in the name...

$$\text{Combinatorics} + \text{Analysis}$$

More specifically, we wish to use the objects and results of *complex* analysis–

What is analytic combinatorics?

It's in the name...

$$\text{Combinatorics} + \text{Analysis}$$

More specifically, we wish to use the objects and results of *complex* analysis–

- Taylor/Laurent series
- Contour integrals
- Residues
- etc.

–to study the *asymptotic* growth behavior of the enumeration function of a combinatorial class.

What is analytic combinatorics?

It's in the name...

$$\text{Combinatorics} + \text{Analysis}$$

More specifically, we wish to use the objects and results of *complex* analysis–

- Taylor/Laurent series
- Contour integrals
- Residues
- etc.

–to study the *asymptotic* growth behavior of the enumeration function of a combinatorial class. Useful for when you don't or can't have a nice closed-form expression for the sequence that you're counting.

Generating functions

Key idea: **Generating functions!**

Generating functions

Key idea: **Generating functions!** If the sequence we are working with is denoted $a(n)$, then we can encode the sequence as the coefficients of a complex power series:

$$\sum_{n=0}^{\infty} a(n) z^n$$

We can then study the sequence "by proxy" using analytic techniques.

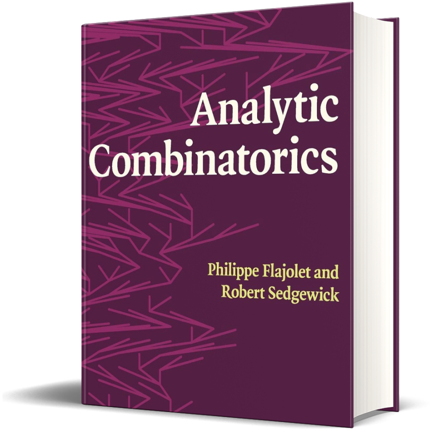
Generating functions

Key idea: **Generating functions!** If the sequence we are working with is denoted $a(n)$, then we can encode the sequence as the coefficients of a complex power series:

$$\sum_{n=0}^{\infty} a(n)z^n$$

We can then study the sequence "by proxy" using analytic techniques. Generally, the nicer the analytic properties of our generating function, the nicer the combinatorial properties of the sequence it represents.

It's well-studied...



What is ACSV?

...but that's only for objects of a single parameter. For *multivariable* sequences it's a bit more complicated.

ACSV (see: acsvproject.com) is an emerging field that utilizes:

- Several complex variables
- Algebra
- Topology
- Morse Theory

to study asymptotics of sequences in *any* number of dimensions.

ACSV

Suppose we're given a rational function in d complex variables:

$$F(\boldsymbol{z}) = \frac{G(\boldsymbol{z})}{H(\boldsymbol{z})}, \qquad H(0) \neq 0$$

Then, F is analytic at the origin with power series:

$$F(\boldsymbol{z}) = \sum_{\boldsymbol{j} \in \mathbb{N}^d} f_{\boldsymbol{j}} \boldsymbol{z}^{\boldsymbol{j}},$$

where the series coefficients $f_{\boldsymbol{j}} = f_{(r_1, r_2, \ldots, r_d)}$ are given by the generalized Cauchy integral formula:

$$f_{\boldsymbol{j}} = \frac{1}{(2\pi i)^d} \int_{T(\boldsymbol{b})} \frac{F(\boldsymbol{z})}{\boldsymbol{z}^{\boldsymbol{j}+\mathbf{1}}} d\boldsymbol{z}, \quad j \in \mathbb{N}^d,$$

$\boldsymbol{b}$ being some point in the polydisk centered at 0 with polyradius $\boldsymbol{a}$.

ACSV

Fix a direction vector $\boldsymbol{r} \in \mathbb{N}^d$. We wish to derive an asymptotic expansion for the sequence $(f_{n\boldsymbol{r}})_{n\in\mathbb{N}}$.

Workflow:

1. Bound $\rho = \limsup_{n\to\infty} |f_{n\boldsymbol{r}}|^{1/n}$, the exponential growth of the coefficients. This is the dominant growth rate of our sequence.
2. Determine the singularities of F that contribute (the most) to our asymptotic behavior.
3. Write down the Cauchy integral for our sequence; "localize" it and compute residues.
4. Apply the Saddle-point method to determine the asymptotic.

Our “Procedure”

INPUT:

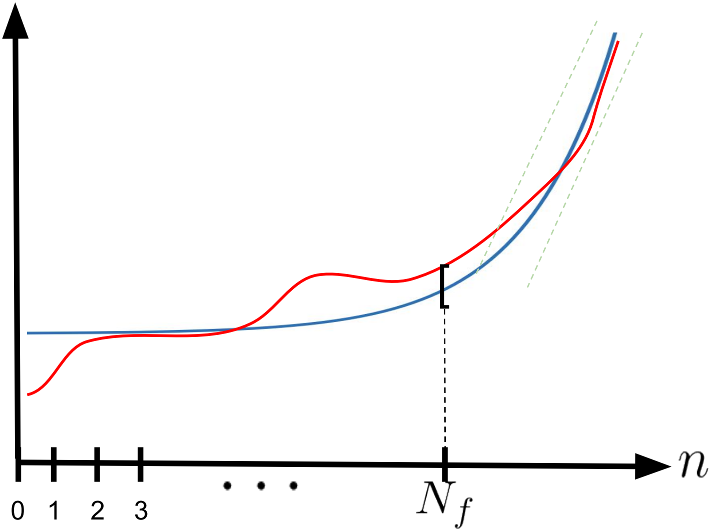
- a rational function $F = G/H$ in d variables
- a direction vector $\boldsymbol{r} \in \mathbb{N}^d$. For this paper, we always take $\boldsymbol{r} = \mathbf{1} := (1, 1, ..., 1)$.

ASSUMING:

- $H(\mathbf{0}) \neq 0$
- $\mathcal{V} = \mathcal{V}(H)$ *smooth*
- F admits a *nondegenrate strictly minimal smooth contributing point* $\boldsymbol{w} \in \mathbb{C}_*^d$.
- The coefficients $f_{n\boldsymbol{r}}$ are eventually positive.

OUTPUT: An $N_f \in \mathbb{N}$ such that $\forall n \geq N_f$, $f_{n\boldsymbol{r}} > 0$.

Diagram...



Our "Procedure"

PROCEDURE:

1. Verify that our set of assumptions hold for our inputs.***
2. Run the smooth ACSV procedure on our inputs, obtaining:
   - a complex multivariate saddle point integral χ which can be viewed as $\chi(n)$ depending on the parameter n.
   - a function $\lambda(n)$ such that $\chi(n)$ (hence $f_{n\mathbf{r}}$) is asymptotically equivalent to $\lambda(n)$.
3. Compute numbers $\tau \in (0, |\mathbf{w}^{-\mathbf{r}}|)$, $c > 0$ such that $\forall n \in \mathbb{Z}_+, |f_{n\mathbf{r}} - \lambda| < c\tau^n$.
4. Show that there exists an $L \in \mathbb{Z}_+$ such that $\forall n \geq L$, $|\chi - \lambda| \leq \lambda - c\tau^n$.
5. This $L = N_f$. Output it.

DONE.

Table of Contents

Bound for family of bivariate functions:

Computation

Let $F : \mathbb{C}^2 \to \mathbb{C}$ *be defined by*

$$F(x, y) = \frac{1}{1 - ax - by + cxy}, \quad a, b \geq c > 1.$$

Put

$$\boldsymbol{w} = (w_1, w_2) := \frac{ab - \sqrt{(ab)^2 - abc}}{ac} \left(1, \frac{a}{b}\right).$$

Then, to prove positivity of the diagonal coefficients of F*'s power series centered at the origin, it suffices to check the first* N_f *terms for positivity.* N_f *is given by the formula...*

Bound for family of bivariate functions:

Computation

$$N_f = \max\left\{N_2, \lceil \delta^{\frac{-1}{\alpha}} \rceil, \left\lceil \left(\frac{c_0}{\mu}\right)^{1/(3\alpha-1)} \right\rceil, \tilde{N}, \left\lceil \left(\frac{c_{16}}{\epsilon}\right)^{1/(3\alpha-1)} \right\rceil, N \right\}$$

where

- $0 < \delta < \min\left\{w_2, \pi/2, \frac{1}{2}\ln(b/(cw_1))\right\}$
- $0 < \epsilon < 1$, $\mu > 0$, *and*
- $\alpha \in \mathbb{Q} \cap (1/3, 1/2)$

...are freely chosen parameters, and constants $N_2, c_0, \tilde{N}, c_{16},$ *and* N *are obtained effectively in the derivation.*

Bound for family of bivariate functions:

In the special case when $a = b$, we also have the following corollary, whose proof is immediate by a 2015 result of Straub:

Corollary

Let F be as described in the above Computation, with $a = b > c > 1$. Then, to prove positivity of all of F's power series coefficients, in any direction, it suffices to check the first N_f terms along the diagonal.

Bound for GRZ function:

Computation

Let $d \geq 4$ be an integer. Let $F_{d!,d} : \mathbb{C}^d \to \mathbb{C}$ be defined by

$$F_{d!,d}(\boldsymbol{z}) = \frac{1}{1 - z_1 \cdots - z_d + d! z_1 \cdot \cdots \cdot z_d}, \quad \forall \boldsymbol{z} \in \mathbb{C}^d.$$

Then, to prove positivity of the diagonal coefficients of $F_{d!,d}$'s power series centered at the origin, it suffices to check the first N_f terms for positivity. N_f is given by the formula...

Bound for GRZ function:

Computation

$$N_f = \max\left\{N_2, N_{16}, \tilde{N}, \left\lceil\left(\frac{c_{18}}{\epsilon}\right)^{1/(3\alpha-1)}\right\rceil, N\right\}$$

with the freely chosen parameters:

- $\delta \in (0, \frac{1}{(d-1)^{d-1}})$, *chosen so that* $\frac{1}{\rho\cdot(d-1)} - 1 \geq |e^{2\delta-1}|$, *($\rho$ being the unique real root of a certain polynomial,*
- $0 < \epsilon, \tilde{\epsilon} < 1$
- $\alpha \in (1/3, 1/2) \cap \mathbb{Q}$
- $\mu > 0$,

along with the constants N_{16}, c_{18}, N, $\tilde{N}$ *and* N_2 *which can be computed (mostly) effectively as described in the derivation.*

First, some elementary lemmas...

Lemma

(Calculation involving the Product Log):
Let $\mathcal{H}$ be a positive real, α a rational strictly between $1/3$ and $1/2$. Then, $\exists \tilde{N} \in \mathbb{Z}_+$, *so that* $\forall n \geq \tilde{N}$, $e^{-\frac{\mathcal{H}}{2} n^{1-2\alpha}} < n^{\frac{1}{2}-3\alpha}$.

Lemma

(Comparing $c\tau^n$ and $\lambda(n)$):
Let $c, D > 0$, $\boldsymbol{w} = (w_1, \ldots w_d) \in (\mathbb{C} - \mathbf{0})^d$ *and* $0 < \tau < |w_1|^{-1} \cdot \ldots \cdot |w_d|^{-1}$. *Then,*
$\exists N \in \mathbb{Z}_+$ *such that* $\forall n \geq N, c\tau^n < D|w_1 \cdot \ldots \cdot w_d|^{-n} n^{(1-d)/2}$.

Both of the claimed quantities we derive formulas for, but they're too lengthy to include here.

Sketch of bound derivations...

STEP 1: Verifying the assumptions. GOAL: Find a *nondegenerate strictly minimal smooth contributing point*:

- Compute the singular variety, $\mathcal{V}$
- Solve the smooth critical equations to obtain the set of *smooth critical* points along the main diagonal, denoted $\mathrm{crit}(\mathbf{1})$:

$$\begin{cases} H(\boldsymbol{w}) = 0 \\ w_1 H_{z_1}(\boldsymbol{w}) - w_j H_{z_j}(\boldsymbol{w}) = 0, \quad 2 \leq j \leq d \end{cases}$$

- Compute *minimal* points lying in $\mathrm{crit}(\mathbf{1})$. It suffices to find $\boldsymbol{w} \in \mathrm{crit}(\mathbf{1})$ such that for no $\boldsymbol{v} \in \mathcal{V}$ is $|\boldsymbol{v}| = t|\boldsymbol{w}|$, for some $t \in (0, 1)$.
- Show that this minimality of $\boldsymbol{w}$ is *strict*.

STEP 1, continued

- Verify that $\boldsymbol{w}$ is *nondegenerate*. This means that the Hessian matrix $\mathcal{H}$ of function ϕ associated with is nonsingular at the origin.

Next, we need to prove that the function F is *eventually* positive:

- For the bivariate case, follows from looking at our “out-of-the-box” asymptotic gotten from running the ACSV procedure:

$$f_{n\mathbf{1}} = \frac{(w_1 w_2)^{-n}}{\sqrt{2\pi n}} \cdot \frac{1}{\sqrt{\det(\mathcal{H})} w_2(b - cw_1)}(1 + O(1/n)),$$

- ...whereas for the GRZ case this is a result of a 2018 paper by Baryshnikov, et al

STEP 2:

Now we introduce the usual objects associated with our analysis:

- $\mathcal{T} := T(\hat{\boldsymbol{\rho}})$, the $(d-1)$-dimensional polytorus centered at $\hat{\boldsymbol{\rho}} \in \mathbb{C}^{d-1}$,
- δ, a positive constant
- $\mathcal{N} = \{(\rho e^{i\theta_1}, \ldots, \rho e^{i\theta_{d-1}}) : \theta_1, \ldots, \theta_{d-1} \in (-\delta, \delta)\}$, our neighborhood of $\boldsymbol{w}$ in $\mathcal{T}$,
- $\mathcal{N}' := \mathcal{T} \setminus \mathcal{N}$, and
- Our analytic parametrization $w = g(\hat{\boldsymbol{z}})$ of $\mathcal{V}$ for $\boldsymbol{z} \in \mathcal{N}$.

...along with Cauchy integral and its localized versions:

$$f_{n\mathbf{1}} = I := \frac{1}{(2\pi i)^d} \int_{\mathcal{T}} \left(\int_{|z_d|=\rho-\delta} F_{d!,d}(\boldsymbol{z}) \frac{dz_d}{z_d^{n+1}} \right) \frac{d\hat{\boldsymbol{z}}}{\hat{\boldsymbol{z}}^{(n+1)\hat{\mathbf{1}}}},$$

$$I_{\text{loc}} := \frac{1}{(2\pi i)^d} \int_{\mathcal{N}} \left(\int_{|z_d|=\rho-\delta} F_{d!,d}(\boldsymbol{z}) \frac{dz_d}{z_d^{n+1}} \right) \frac{d\hat{\boldsymbol{z}}}{\hat{\boldsymbol{z}}^{(n+1)\hat{\mathbf{1}}}},$$

$$I_{\text{out}} := \frac{1}{(2\pi i)^d} \int_{\mathcal{N}} \left(\int_{|z_d|=\rho+\delta} F_{d!,d}(\boldsymbol{z}) \frac{dz_d}{z_d^{n+1}} \right) \frac{d\hat{\boldsymbol{z}}}{\hat{\boldsymbol{z}}^{(n+1)\hat{\mathbf{1}}}}, \text{ and}$$

$$\chi := I_{\text{loc}} - I_{\text{out}} =$$

$$\frac{-1}{(2\pi i)^d} \int_{\mathcal{N}} \left(\int_{|z_d|=\rho+\delta} F_{d!,d}(\boldsymbol{z}) \frac{dz_d}{z_d^{n+1}} - \int_{|z_d|=\rho-\delta} F_{d!,d}(\boldsymbol{z}) \frac{dz_d}{z_d^{n+1}} \right) \frac{d\hat{\boldsymbol{z}}}{\hat{\boldsymbol{z}}^{(n+1)\hat{\mathbf{1}}}},$$

STEP 3:

$$\chi = \frac{1}{(2\pi i)^{d-1}} \int_{\mathcal{N}} \frac{(1 - d! z_1 \dots z_{d-1})^n}{z_1^{n+1} \dots z_{d-1}^{n+1} (1 - z_1 - \dots - z_{d-1})^{n+1}} d\hat{\boldsymbol{z}}.$$

SKIP!

STEP 4:

Skippity skip skipped

Table of Contents

Results... practical?

- Bivariate Case: For example, with $a = 3, b = 4, c = 3$, and the following choice of parameters:
  - $\delta = \frac{1}{2}\min(w_2, \pi/2, \log(\sqrt{\frac{b}{cw_1}})) = w_2/2$
  - $\epsilon = 2/5$
  - $\alpha = 639/1280$
  - $\mu = 1/2$

  we end up with $\mathbf{N_f} = 1269$. Positivity up to 1269 is easy to check on my home computer.

Results... practical?

- Bivariate Case: For example, with $a = 3, b = 4, c = 3$, and the following choice of parameters:
    - $\delta = \frac{1}{2}\min(w_2, \pi/2, \log(\sqrt{\frac{b}{cw_1}})) = w_2/2$
    - $\epsilon = 2/5$
    - $\alpha = 639/1280$
    - $\mu = 1/2$

  we end up with $\mathbf{N_f} = 1269$. Positivity up to 1269 is easy to check on my home computer.
- GRZ case: Even for $d = 4$, with a very reasonable choice of parameters we end up with

$$N_f \geq 4547349426.$$

...so checking positivity up to N_f not too practical.

Future Questions / Directions

- N_f can be improved, to a point, by judicious choice of parameters $(\delta, \epsilon, \alpha, \mu)$. Find optimal choice of parameters and "limit value" of N_f as we approach this point in parameter space.
- Obvious improvements in our analysis. Bounds can be tighter.
- Solve issue with reduction to optimization problem over Torus in Step 3 of GRZ case.
- Automate the analysis in steps 3 and 4.
- Apply this process to other functions in the literature.
- Examine degenerate cases, e.g. when our coefficients are eventually zero in a direction, find N_f guaranteeing henceforward "zero-ness."
- Expand methodology to general "non-smooth" case of ACSV: poles on hyperplane arrangements and beyond!

Thank you!

Thank you! :)