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Explicit Diagonal Asymptotics of Symmetric Multi-Affine Rational Functions via ACSV

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AARMS CMS SMC

Diagonal Asymptotics and Positivity

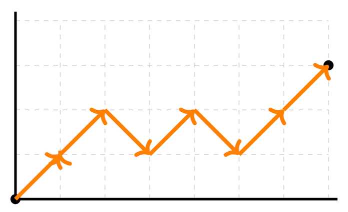
Problem

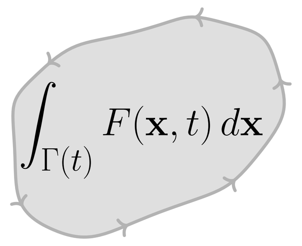
Let $\boldsymbol{z} = (z_1, \dots z_d)$ be complex variables, $F(\boldsymbol{z}) = \frac{G(\boldsymbol{z})}{H(\boldsymbol{z})}$ rational with G and H coprime, $H(\mathbf{0}) \neq 0$, and $\boldsymbol{r} \in \mathbb{N}^d$. Letting

$$F(\boldsymbol{z}) = \sum_{\boldsymbol{i} \in \mathbb{N}^d} f_{\boldsymbol{i}} \boldsymbol{z}^{\boldsymbol{i}}$$

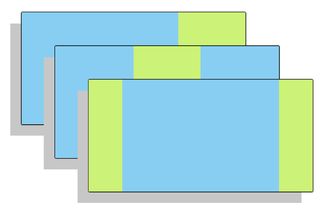
denote F's power series representation at the origin, compute asymptotics for the *diagonal* coefficient sequence $(f_{n\boldsymbol{r}})$.

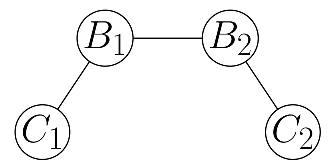
Applications: Diagonals of rational functions appear as counting sequences for...


Lattice paths



Period Integrals


Irrational Tilings



Graphs & Networks

...

A Special Application – Positivity

Problem

- **Total Positivity:** Given F, are all $f_{\boldsymbol{i}}$ positive?
- **Directional Positivity:** Given $\boldsymbol{r}$, are all $f_{n\boldsymbol{r}}$ positive?
- **Eventual Positivity:** Are the $f_{\boldsymbol{i}}$ (or $f_{n\boldsymbol{r}}$) *eventually* positive?

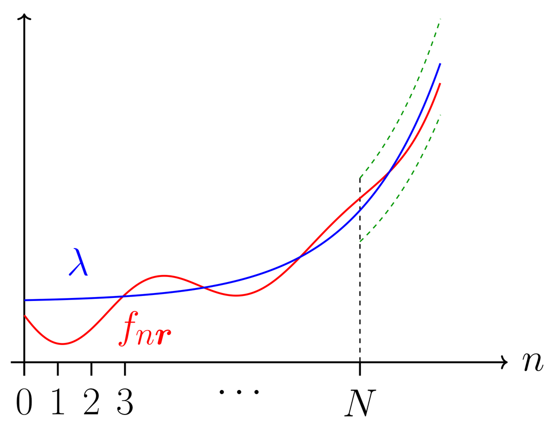
Examples: The following functions are totally positive.

- $\dfrac{1}{1-x-y-z+4xyz}$
- $\dfrac{1}{e_2(1-x, 1-y, 1-z, 1-w)}$
- $\dfrac{1}{e_3(1-x, 1-y, 1-z, 1-w)}$
- $\dfrac{1}{1-x_1-\cdots-x_d+d!x_1\cdots x_d}$

Proving Directional Positivity

To show $f_{n\boldsymbol{r}} > 0$ for *all* n:

1. Derive asymptotic $\lambda(n)$ for $f_{n\boldsymbol{r}}$.
2. Show λ is positive.
3. Bound $f_{n\boldsymbol{r}}$ close to λ, explicitly.
4. Verify positivity for finitely many terms.



Related Work

Inspired by [Baryshnikov-Melczer-Pemantle-Straub 2018]. There is also the survey [Straub-Zudilin 2015], and the following **software:**

Sage ACSV

https://acsvmath.github.io/sage_acsv/

Ore Algebras

See the `bound_coefficients` command in the `Ore_Algebra` package, or [Dong-Melczer-Mezzarobba 2023].

Explicit Asymptotics

Consider $H(\boldsymbol{z})$ *symmetric* (invariant under variable permutation) and *multi-affine* (degree 1 in each variable), so

$$F(\boldsymbol{z}) = \frac{G(\boldsymbol{z})}{1 - \sum_{k=1}^d a_k e_k(\boldsymbol{z})},$$

the a_k real and $e_k(\boldsymbol{z})$ the kth elementary symmetric polynomial in $\boldsymbol{z}$. Fix $\boldsymbol{r} = (\mathbf{1}, \mathbf{1}, \dots, \mathbf{1})$. Points in $\mathbb{C}^d$ contributing to asymptotics of $(f_{n\boldsymbol{r}})$ satisfy

$$\begin{aligned} H(\boldsymbol{w}) &= 0 \\ H_{z_j}(\boldsymbol{w}) &\neq 0 \quad \text{for some } j \\ w_1 H_{z_1}(\boldsymbol{w}) - w_j H_{z_j}(\boldsymbol{w}) &= 0 \quad (2 \leq j \leq d). \end{aligned} \qquad (\star)$$

Notation

- $\delta^H(t) = H(t, t, \dots, t)$.
- ρ root of δ^H with minimal modulus.
- $\boldsymbol{w}_0 = (\rho, \rho, \dots, \rho) = \rho\mathbf{1}$, which always contributes via Grace-Walsh-Szegő.
- $E = \{$solutions to $(\star)$ with minimal coordinate-wise modulus$\}$.
- $U = \dfrac{\rho H_{z_i z_j}(\boldsymbol{w}_0)}{H_{z_d}(\boldsymbol{w}_0)}$.
- $\mathcal{H} = (1-U)(I + \mathbf{1}\mathbf{1}^T)$, the Hessian of ϕ at $\mathbf{0}$, as seen in the next column.

Main theorem – Explicit Asymptotic Bounds

Let $F, \boldsymbol{r}, \rho$ satisfy the constraints of the table below. Putting

$$\lambda(n) = \rho^{-(d+1)n} n^{(1-d)/2} \cdot \frac{(2\pi d(1-U))^{(1-d)/2}}{a_1 + \sum_{k=2}^d a_k \binom{d-1}{k-1} \rho^{k-1}},$$

λ is positive, and there exists a computable constant N such that $|f_{n\boldsymbol{r}} - \lambda(n)| < \frac{1}{2}\lambda(n)$ for all $n \geq N$.

Assumption:	Needed so that:
H irreducible	$\mathcal{V} = \{\boldsymbol{z} : H(\boldsymbol{z}) = 0\}$
$E = \{\boldsymbol{w}_0\}$	$\mathcal{H} = \mathcal{H}_{\boldsymbol{w}}$ is uniform, asymptotic explicit
$H_{z_d}(\boldsymbol{w}_0) \neq 0$	$\mathcal{V}$ is smooth; (also works with $H_{z_j}(\boldsymbol{w}_0) \neq 0$)
$U < 1$	$\det \mathcal{H} \neq 0$, $\lambda > 0$
$A(\mathbf{0}) > 0$	$\lambda > 0$
$\rho > 0$	$\lambda > 0$

Corollary: To ensure $(f_{n\boldsymbol{r}})$ is positive it suffices to check the terms $f_{\boldsymbol{r}}, f_{2\boldsymbol{r}}, \cdots f_{(N-1)\boldsymbol{r}}$.

Examples

Bivariate

$F(x, y) = \dfrac{1}{1 - ax - by + cxy}$ $\quad a, b, c \geq 0,\ ab > c$. With $a = c = 3,\ b = 4$ we get $\boldsymbol{N = 1269}$.

Gillis-Reznick-Zeilberger

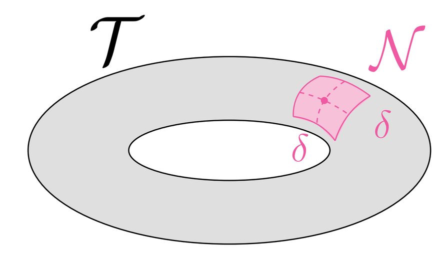
$F(z_1, \dots, z_d) = \dfrac{1}{1 - \sum_i z_i + d! \prod_i z_i}$, $\quad d \geq 4$. With $d = 4$ we get $\boldsymbol{N > 10^6}$.

Proof Idea

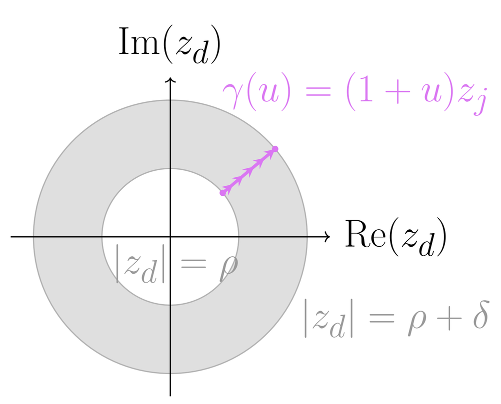
Start with:

$$f_{n\boldsymbol{r}} = \frac{1}{(2\pi \mathrm{i})^d} \int_{T(\boldsymbol{y})} F(\mathbf{z}) \frac{d\boldsymbol{z}}{\boldsymbol{z}^{n\boldsymbol{r}+\mathbf{1}}}, \qquad \boldsymbol{y} \text{ in } F\text{'s domain of convergence.}$$

Pick the point $\boldsymbol{w}_0$, localize around it:

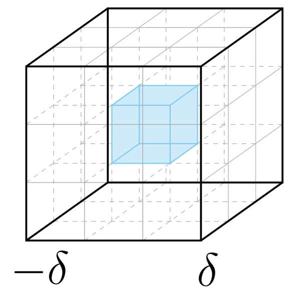


$$\begin{aligned} f_{n\boldsymbol{r}} = I &= \frac{1}{(2\pi \mathrm{i})^d} \int_{\mathcal{T}} \left(\int_{|z_j| = \rho - \delta} F(\mathbf{z}) \frac{dz_j}{z_j^{nr_j+1}} \right) \frac{d\widehat{\boldsymbol{z}}_j}{\widehat{\boldsymbol{z}}_j^{n\widehat{\boldsymbol{r}}_j + \mathbf{1}}} \\ I^{\text{loc}} &= \frac{1}{(2\pi \mathrm{i})^d} \int_{\mathcal{N}} \left(\int_{|z_j| = \rho - \delta} F(\mathbf{z}) \frac{dz_j}{z_j^{nr_j+1}} \right) \frac{d\widehat{\boldsymbol{z}}_j}{\widehat{\boldsymbol{z}}_j^{n\widehat{\boldsymbol{r}}_j + \mathbf{1}}} \\ I^{\text{out}} &= \frac{1}{(2\pi \mathrm{i})^d} \int_{\mathcal{N}} \left(\int_{|z_j| = \rho + \delta} F(\mathbf{z}) \frac{dz_j}{z_j^{nr_j+1}} \right) \frac{d\widehat{\boldsymbol{z}}_j}{\widehat{\boldsymbol{z}}_j^{n\widehat{\boldsymbol{r}}_j + \mathbf{1}}} \\ \chi &= I^{\text{loc}} - I^{\text{out}} \end{aligned}$$

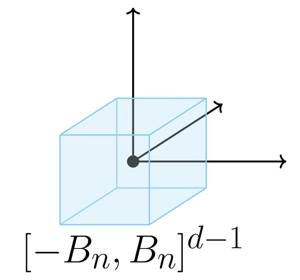


- Can show $|I - I^{\text{loc}}|, \quad |I^{\text{out}}| < c\tau^n < \rho^{-dn}$.
- So, $|f_{n\boldsymbol{r}} - \chi| \leq |I - I^{\text{loc}}| + |I^{\text{out}}| = O(\tau^n)$ for some $\tau < \rho^{-dn}$.

Parametrize $\mathcal{N}$, write χ as saddle point integral: $\chi = \dfrac{\boldsymbol{w}^{-n\mathbf{r}}}{(2\pi)^{d-1}} \displaystyle\int_{[-\delta,\delta]^{d-1}} A(\boldsymbol{\theta}) e^{-n\phi(\boldsymbol{\theta})} d\boldsymbol{\theta}$



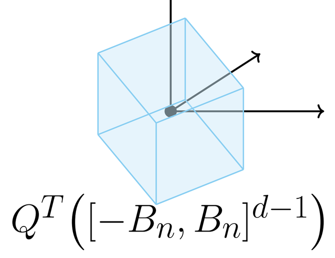
$$\chi = \frac{A(\mathbf{0})\boldsymbol{w}^{-n\mathbf{r}}}{(2\pi)^{d-1}} \left(\underbrace{\int_{[-B_n, B_n]^{d-1}} e^{-\frac{n}{2}\boldsymbol{\theta}^T \mathcal{H} \boldsymbol{\theta}} d\boldsymbol{\theta}} + \underbrace{\int_{\text{outer regions}} e^{-\frac{n}{2}\boldsymbol{\theta}^T \mathcal{H} \boldsymbol{\theta}} d\boldsymbol{\theta}}_{= O(\text{error smaller than } \rho^{-dn})} \right)$$



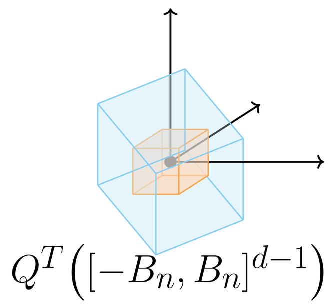
$$\int_{[-B_n, B_n]^{d-1}} e^{-\frac{n}{2}\boldsymbol{\theta}^T \mathcal{H} \boldsymbol{\theta}} d\boldsymbol{\theta}$$

Spectral Theorem

$\mathcal{H} = QDQ^T$, with Q, D an eigen.-decomp. for $\mathcal{H}$



$$= \int_{Q^T([-B_n, B_n]^{d-1})} e^{-\frac{n}{2}\left(\lambda_1 \theta_1^2 + \cdots + \lambda_{d-1}\theta_{d-1}^2\right)} d\boldsymbol{\theta}$$



$$= \int_{\left[\frac{-B_n}{\sqrt{d-1}}, \frac{B_n}{\sqrt{d-1}}\right]^{d-1}} e^{-\frac{n}{2}\sum_i \lambda_i \theta_i^2} d\boldsymbol{\theta} + O(\text{small error}).$$

$$= \int_{\mathbb{R}^{d-1}} e^{-\frac{n}{2}\sum_i \lambda_i \theta_i^2} d\boldsymbol{\theta} + O(\text{small error}).$$

$$\implies \chi = \lambda(1 + O(n^{-\epsilon})).$$

Combining:

$$\begin{aligned} |f_{n\boldsymbol{r}} - \lambda| &\leq |f_{n\boldsymbol{r}} - \chi| + |\chi - \lambda| \\ &< c\tau^n + \lambda n^{-\epsilon} \leq \frac{1}{2}\lambda, \quad \text{as needed.} \quad \square \end{aligned}$$