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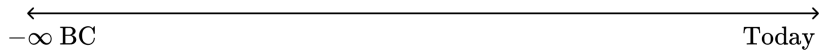


Algorithms for Analytic Combinatorics: Positivity Bounds and D-finite Operators

Advisors: Stephen Melczer and Rafael Oliveira

Master's Thesis Presentation

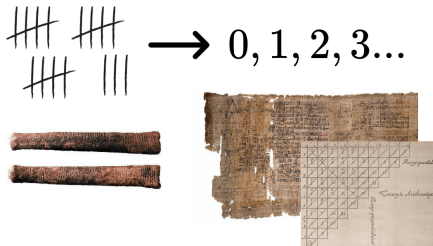
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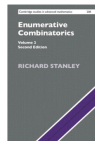


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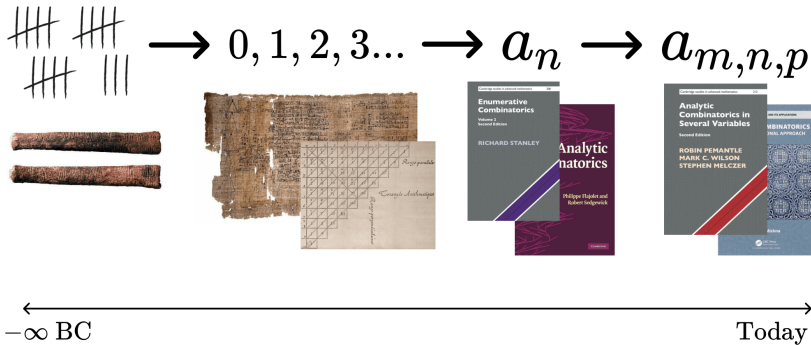
A 30-second history of Combinatorics...

 $\rightarrow 0, 1, 2, 3... \rightarrow a_n$



$\leftarrow \infty \text{ BC} \quad \quad \quad \text{Today} \rightarrow$

A 30-second history of Combinatorics...



Analytic Combinatorics: What and Why?

What: Given (a_n) , use analytic properties of **generating function (GF)**

$$A(z) = \sum_{n \geq 0} a_n z^n \text{ to derive facts about } (a_n).$$

Why: Because it may work (asymptotically) when other methods fail.

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Ways of “chopping” $[n]$ into parts and listing the parts. E.g. $(\{1, 2\}, \{3\})$, $(\{3\}, \{1, 2\})$ and $(\{1, 2, 3\})$ are some ordered partitions of $[3]$.

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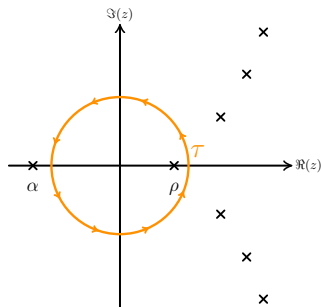
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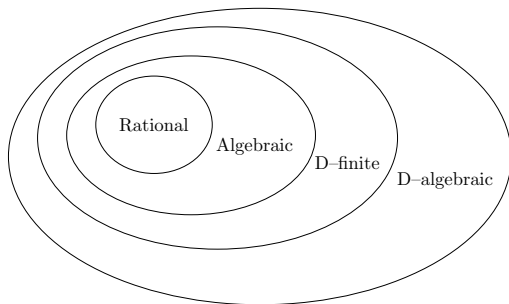
to show that

$$a_n \sim \frac{n!}{(1 + \rho)\rho^{n+1}} \quad \text{where} \quad \rho \approx 1.1461932206.$$



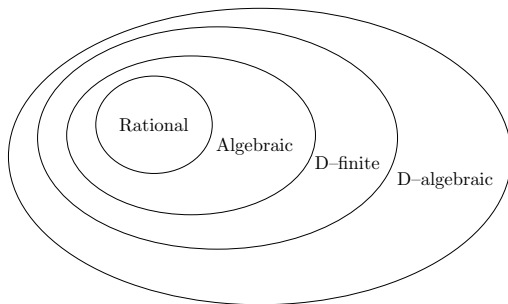
The Hierarchy of GF types...

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Definition

F is called **D-finite** if there exists a positive integer d and polynomials $p_0(z), p_1(z), \dots, p_d(z) \in \mathbb{C}[z]$ such that

$$p_d(z)F^{(d)}(z) + \dots + p_2(z)F''(z) + p_1(z)F'(z) + p_0(z)F(z) = 0.$$

Another important GF class

Let

- $\mathbf{z} = (z_1, \dots, z_d)$ be complex variables,
- $$F(\mathbf{z}) = \sum_{i_1, i_2, \dots, i_d \geq 0} f_{i_1, i_2, \dots, i_d} z_1^{i_1} z_2^{i_2} \cdots z_d^{i_d} = \sum_{\mathbf{i} \in \mathbb{N}^d} f_{\mathbf{i}} z^{\mathbf{i}},$$
- $\mathbf{r} = (r_1, \dots, r_d) \in \mathbb{Z}_+^d$.

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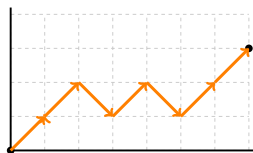
Given a multivariate power series $F(\mathbf{z})$ and a direction $\mathbf{r}$, the **$\mathbf{r}$ -diagonal** of F is

$$(\Delta_{\mathbf{r}} F)(t) = \sum_{n=0}^{\infty} f_{n\mathbf{r}} t^n.$$

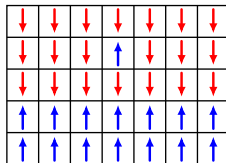
When $\mathbf{r} = \mathbf{1}$ we call $\Delta_{\mathbf{r}} F$ the **main diagonal** and denote it by $(\Delta F)(t)$.

Diagonals appear in many places...

Univariate GFs which are multivariate rational diagonals appear in the study of:

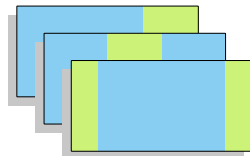


Lattice paths

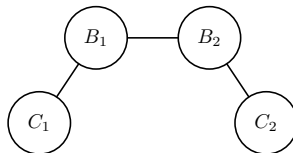


Ising Model

...etc.



Irrational tilings



Graphs and Networks

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Let $R(z, t)$ be rational and $\gamma \subset \mathbb{C}^d$ a deformation of a small product of circles about 0 . The **period** of R with respect to γ is the integral

$$I(t) = \oint_{\gamma} R(z, t) dz.$$

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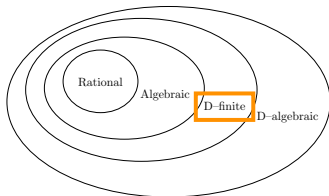
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Thus, diagonals belong in our hierarchy...

To study $\Delta_r F$, we study linear differential operators $\mathcal{L}$ such that $\mathcal{L}(\Delta_r F) = 0$.



Project 1:
Explicit Error Bounds for ACSV

Project 2:
SagePeriods Package

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- Implement algorithm of Lairez for computing rational periods in Sage.
- Apply implementation to computing diagonals of rational functions.

Given $F(\mathbf{z}) = \sum_{\mathbf{i} \in \mathbb{N}^d} f_{\mathbf{i}} \mathbf{z}^{\mathbf{i}}$ and $\mathbf{r} \in \mathbb{Z}_+$, write $F = G/H$ for polys G and H with $\gcd(G, H) = 1$ and $H(\mathbf{0}) \neq 0$.

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Question:

For a given $M \in \mathbb{Z}_+$, how do we compute $\lambda_M(n)$ such that

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- **Step 4:** Find **minimal critical** points.

Roughly speaking,

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A minimal critical point σ is called

- **strictly minimal** if $\{H(z) = 0\} \cap T(\sigma) = \{\sigma\}$,
- **finitely minimal** if $\{H(z) = 0\} \cap T(\sigma)$ is finite.

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ACSV Background

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Note:

$$\chi = \frac{\sigma^{-nr}}{(2\pi)^{d-1}} \underbrace{\int_{[-\delta, \delta]^{d-1}} A(\theta) e^{-n\phi(\theta)} d\theta}_{\text{sub-exponential factor}}$$

But in order to evaluate χ , we need the following.

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After evaluation of χ , we get the following:

Theorem [Pemantle-Melczer-Wilson '23]

Suppose that σ is as above, and such that $H_{z_j}(\sigma) \neq 0$ for some $j \in \{1, \dots, d\}$. Then for any nonnegative integer M there exist computable constants $C_0, \dots, C_M$ and $\mathcal{H}$ such that

$$f_{nr} = \sigma^{-nr} n^{(1-d)/2} \frac{(2\pi)^{(1-d)/2}}{\sqrt{\det(r_j \mathcal{H})}} \left(\sum_{k=0}^M C_k (r_j n)^{-k} + O\left(n^{-M-1}\right) \right).$$

How do we turn this into bounds?

Recall:

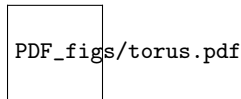
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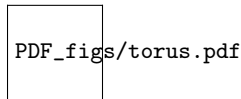
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All three steps in this process require some approximation, hence require “explicit” versions.

Step 5: “Localizing to $\mathcal{N}$ ” ...

We must construct a δ defining a torus neighborhood

$$\mathcal{N} = \{\hat{\sigma} e^{i\theta} : \theta \in [-\delta, \delta]^{d-1}\}$$

so that:

1. We can parametrize z_j in our singular variety $\{H(z) = 0\}$ i.t.o. $\hat{z} \in \mathcal{N}$.
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Lemma [Melczer-S '26+]

Let σ be strictly minimal critical. Then we can construct a $\delta > 0$ defining $\mathcal{N} = \{\hat{\sigma}e^{i\theta} : \theta \in [-\delta, \delta]^{d-1}\}$ and a function $g : \mathcal{N} \rightarrow \mathbb{C}$ accomplishing 1.

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Lemma [Melczer-S '26+]

We can restrict δ so that for each fixed $\hat{z} \in \mathcal{T}$, there is at most one pole of F when $|z_j| \in [|\sigma_j| - \delta, |\sigma_j| + \delta]$. If $\hat{z} \in \mathcal{N}$ this pole is $g(\hat{z})$.

Step 6: “Reduce to residue integral...”

Doing that and applying univariate residue theorem, we construct an integral

$$\chi = \frac{\sigma^{-nr}}{(2\pi)^{d-1}} \int_{[-\delta, \delta]^{d-1}} A(\boldsymbol{\theta}) e^{-n\phi(\boldsymbol{\theta})} d\boldsymbol{\theta} ,$$

with $A(\boldsymbol{\theta}) =$ rational function in $\widehat{\boldsymbol{\sigma}} e^{i\boldsymbol{\theta}}$ and $g\left(\widehat{\boldsymbol{\sigma}} e^{i\boldsymbol{\theta}}\right)$

and $\phi(\boldsymbol{\theta}) = \log \frac{g\left(\widehat{\boldsymbol{\sigma}} e^{i\boldsymbol{\theta}}\right)}{g(\widehat{\boldsymbol{\sigma}})} + i(\widehat{\boldsymbol{r}} \cdot \boldsymbol{\theta})$.

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Need to show χ is “good” approximation for f_{nr} .

Proposition [Melczer-S '26+]

If F admits some strictly-minimal smooth critical point σ in the direction r , one can construct a $c > 0$ and $0 < \tau < |\sigma|^{-r}$ so that for all n ,

$$|f_{nr} - \chi| < c\tau^n.$$

Step 7: “Evaluate saddle point integral χ .”

After taking residues, we get
$$\chi = \frac{\sigma^{-nr}}{(2\pi)^{d-1}} \underbrace{\int_{[-\delta, \delta]^{d-1}} A(\theta) e^{-n\phi(\theta)} d\theta}_{I(n)},$$

with $A(\theta) =$ rational function in $\hat{\sigma}e^{i\theta}$ and $g(\hat{\sigma}e^{i\theta})$

and
$$\phi(\theta) = \log \frac{g(\hat{\sigma}e^{i\theta})}{g(\hat{\sigma})} + i(\hat{r} \cdot \theta).$$

Proposition [Melczer-S '26+]

Let σ be as above, and also nondegenerate. For any $M \in \mathbb{Z}_+$ the integral $I(n)$ has asymptotic expansion

$$S_M(n) = n^{-(d-1)/2} \frac{(2\pi)^{(d-1)/2}}{\det(r_j \mathcal{H})^{1/2}} \sum_{\ell=0}^{M-1} C_\ell n^{-\ell},$$

for explicit constants $\mathcal{H}$ and C_ℓ . Moreover, there is an explicit error $E_M(n) = O\left(n^{-(d-1)/2-M}\right)$ so that $|I(n) - S_M(n)| \leq E_M(n)$.

The result...

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Theorem [Melczer-S '26+]

Let $F, \mathbf{r}, j, \boldsymbol{\sigma}$ be as above. For any $M \in \mathbb{Z}_+$ there is an asymptotic expansion

$$\lambda_M(n) = \frac{\boldsymbol{\sigma}^{-n\mathbf{r}}}{(2\pi)^{(d-1)/2}} n^{(1-d)/2} (\det \mathcal{H})^{-1/2} \sum_{\ell=0}^{M-1} C_\ell n^{-\ell},$$

an error expansion $E_M(n) = O\left(|\boldsymbol{\sigma}|^{-n\mathbf{r}} n^{(1-d)/2-M}\right)$, and an integer N so that for all $n \geq N$, $|f_{n\mathbf{r}} - \lambda_M(n)| \leq E_M(n)$.

The result...

Theorem [Melczer-S '26+]

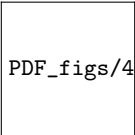
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This can be generalized to the cases where:

- F has a higher-order pole at σ .
- σ is *finitely* minimal.



PDF_figs/499_illustration.pdf

If $\lambda = \lambda_M(n) > 0$ we have the following:

Corollary

(f_{nr}) is positive if $f_0, f_r, f_{2r}, \dots, f_{(N-1)r}$ are positive.

Used similar ideas to [Baryshnikov et al. '18] to derive a couple function classes where leading asymptotic λ_0 is positive.

Applications of Positivity:

- Discretizations of PDEs [Szegő 1922]
- Existence & Uniqueness of solution for Canham model of biomembranes [Melczer-Mezzarobba '23]

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Examples: The following functions are totally positive.

- $\frac{1}{1 - x - y - z + 4xyz}$
- $\frac{1}{e_2(1 - x, 1 - y, 1 - z, 1 - w)}$
- $\frac{1}{e_3(1 - x, 1 - y, 1 - z, 1 - w)}$
- $\frac{1}{1 - x_1 - \dots - x_d + d!x_1 \dots x_d}$

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Recent work:

- *On Positivity and Minimality for Second-Order Holonomic Sequences* [Kenison et al. '20]
- *Positivity Proofs for Linear Recurrences through Contracted Cones* [Ibrahim-Salvy '24]
- *Positivity Proofs for Linear Recurrences with Several Dominant Eigenvalues* [Ibrahim '25]

Switching gears...

Recall...

Fact

Given $F(z)$, its r -diagonal $\Delta_r F$ is a period, hence is D-finite.

So, given $\Delta_r F$ there exists $\mathcal{L}$ such that $\mathcal{L}(\Delta_r F) = 0$.

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Software for computing $\mathcal{L}$:

- Mgfund by F. Chyzak (2000), for Maple
- HolonomicFunctions by C. Koutschan (2010), for Mathematica
- Periods by P. Lairez (2014), for MAGMA
- MultivariateCreativeTelescoping by H. Brochet (2026), for Julia

Our implementation is

- based off of Lairez' algo & MAGMA package.
- designed for compatibility with:
 - **ore_algebra** by M. Kauers et al. (2013)
 - **sage_acsv** by A. Luo et al., (2023)
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```
sage: from sage_periods import compute_diagonal_annihilator as diagonal
sage: var('w x y z u')
sage: diagonal(1/(1-x-y))
> (t - 1/4)*Dt + 1/2
```

```
compute_diagonal_annihilator(R, r=None, vari=None, Dt=None, t=None)

compute_period_annihilator(R, t, Dt)
```

Periods Background

Given $I(t) = \int_{\gamma} R(z, t) dz$, **how to compute** $\mathcal{L}$ **such that** $\mathcal{L}I = 0$?

Write $R = a/f^q$ with a, f homogeneous polys, f squarefree, and $q \in \mathbb{Z}_+$,

$\mathcal{R}_f = \{\text{rational functions of the form } b/f^\ell\}.$

Then, $\mathcal{L}R = \sum_{i=0}^d \partial_i(B_i)$ for $B_i \in \mathcal{R}_f \implies \mathcal{L}\left(\int_{\gamma} R(z, t) dz\right) = 0.$

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Idea

Find $[\cdot] : \mathcal{R}_f \rightarrow \mathcal{R}_f$ which

1. Is $\mathbb{C}(t)$ -linear
2. Has $[R] = 0 \iff R = \sum_i \partial_i(C_i)$, $C_i \in \mathcal{R}_f$
3. Has all $[\partial_t^k R]$, $k \geq 0$, contained in finite-dimensional vector space.

Then, for large enough m ,

$$\sum_{k=0}^m a_k(t) [\partial_t^k R] = 0 \implies \mathcal{L} = \sum_{k=0}^m a_k(t) \partial_t^k \text{ works.}$$

The **Griffiths-Dwork reduction**, denoted $[\cdot]_{GD}$, uses a Groebner basis for $\text{Jac}(f) = (\partial_0 f, \partial_1 f, \dots, \partial_d f)$ to:

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- Write
$$R = \frac{r}{f^q} + \underbrace{\frac{\frac{1}{q-1} \sum_i \partial_i b_i}{f^{q-1}}}_{R_1} + \sum_i \partial_i \left(\frac{-b_i}{(q-1)f^{q-1}} \right)$$

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Theorem [Dwork '62, '64, Griffiths '69]

When $\text{Jac}(f)$ is smooth in $\mathbb{P}^d$, this works! (I.e., $[\cdot]_{GD}$ satisfies properties 1 - 3).

How do we extend to the nonsmooth case?

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Theorem [Lairez '16]

One can construct spaces W_q^r which capture nonsmooth relations and stabilize for r large enough. Gaussian elimination modulo W_q^r gives a “good” reduction.

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So, we have an algorithm.

Conjecture [Dimca '91, Lairez '16]

Let r_F be the minimal r for which the spaces W_q^r stabilize. Then, $r_F \leq n + 1$.

Proving the conjecture would allow for an algorithmic analysis.

- Improved reduction $[\cdot]_r'$ from Lairez, using dimension-reduced spaces X_q^r .
- Modular computation: $\mathbb{C}(t) \xrightarrow{\text{eval}} \mathbb{F}_p \xrightarrow{\text{interp}} \mathbb{F}_p(t) \xrightarrow{\text{CRT}} \mathbb{C}(t)$.
- Handling of non-main diagonals via *root of unity filter*.
- Full API documentation plus webpage:

 Sage Periods

 Q Search C791+K

 sage-periods: A SageMath Package to Compute Rational Diagonals and Periods

 Reference Manual

picard_fuchs

compute_diagonal_annihilator

compute_period_annihilator

compute_homogenization

compute_prepared_fraction

compute_gauss_manin_connection

rham_koszul

RhamKoszulData

__init__

RKGaussManinData

basis_szyzgies

exterior_derivative

elementary_reduction_step

basis_U

hom_reduce

Reference Manual

sage_periods.picard_fuchs

Main functions to compute annihilating D-finite operators for diagonals and periods.

compute_diagonal_annihilator(R, r=None, var=None, Dt=None, t=None)

Given a symbolic rational function $R(x_1, \dots, x_d)$, compute a D-finite equation annihilating the r -diagonal of R .

INPUT:

- R -- A rational function, either an element of `SymbolicRing` or the Fraction Field of a Multivariate Polynomial Ring.
- r -- (Optional) A vector of integers specifying the direction of the diagonal, taken as the all ones vector if not specified.
- var -- (Optional) A vector of all variables appearing in R , used to fix the order of the variables for computation. Taken as `R.variables()` if not specified.
- Dt -- (Optional) The name for the differential symbol in the output, taken as `Dt` by default.
- t -- (Optional) The name for the variable in the output, taken as `t` by default.

OUTPUT:

- L -- An element of the differential `DifferentialAlgebra` in `t` and `Dt` that annihilates the r -diagonal of R or, if `DifferentialAlgebra` is not available, an element of the `DifferentialPolynomialRing` over `QQ[t]` with derivation symbol `Dt`.

EXAMPLES:

SagePeriods:

- Computation of certificates
- Heuristics:
 - diagonal pivot entry
 - toric COV,
 - variable order
- Syzygy profiling
- Smaller operators using algorithmic guessing
- Interfaces to & parallel implementations for other software (M2)
- Multiprocessing
- Automate computation of connection coefficients
- Algebraic periods
- Incorporation into `sage_acsv`

ACSV Error Bounds:

- (Re)-implement bounds in Sage
- Test practicability on examples
- Use bounds & code to solve open positivity problems in the literature.
- Compare to other methods of proving positivity:
 - Cylindrical Algebraic Decomposition
 - Contracted Cones
 - D-finite approach

The End



Thank you!