EFFECTIVE COMPUTATION OF ASYMPTOTIC
POSITIVITY-GUARANTEEING INDEX USING ACSV

BY

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ABSTRACT

We investigate the possibility of determining positivity of the power series coefficients for multivariate rational functions algorithmically, using the tools of analytic combinatorics in several variables. Specifically, we demonstrate a method for proving all coefficients of a series diagonal are positive, given the knowledge that they are *eventually* positive. We do this by first computing an asymptotic expansion using ACSV, then constructing an explicit upper bound for the index at which the positive leading term of the asymptotic will dominate all other error terms.

We apply this procedure to a novel family of bivariate rational functions, as well as a family of functions discussed by Gillis, Reznick, and Zeilberger in their 1983 paper [1], re-proving a conjecture of theirs (with some caveats).

While limited in scope to these two examples, the process of replacing Big-O terms with explicit error bounds in the course of our analysis acts as a proof of concept for proving complete positivity of power series coefficients, and strongly suggests the possibility of automated verification of positivity for at least certain classes of rational multivariate functions.

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LIST OF SYMBOLS

Here we list some basic notions, conventions and notations used throughout.

$\mathbb{N}$	$\{0, 1, 2, 3, ...\}$, the set of natural numbers
$\mathbb{Z}_+$	$\mathbb{N} - \{0\}$, the set of positive integers
[n	] The set $\{1, 2, 3, ..., n\}$
$\mathbb{C}_*$	$\mathbb{C} - \{0\}$
$\boldsymbol{z}$	The vector $(z_1, ..., z_d) \in \mathbb{C}^d$ of d complex variables
$\boldsymbol{z}_{\hat{k}}$	Shorthand for $(z_1, ..., z_{k-1}, z_{k+1}, ..., z_d) \in \mathbb{C}^{d-1}$ obtained by removing the kth entry of $\boldsymbol{z}$
$\hat{\boldsymbol{z}}$	Shorthand for $\boldsymbol{z}_{\hat{d}}$
$d\boldsymbol{z}$	Shorthand for $dz_1 dz_2 ... dz_d$
$\boldsymbol{z}^{\boldsymbol{i}}$	Shorthand for monomial $z_1^{i_1} z_2^{i_2} ... z_d^{i_d}$
$\lvert\boldsymbol{z}\rvert$	The point $(\lvert z_1\rvert, ..., \lvert z_d\rvert) \in \mathbb{R}^d_{\geq 0}$ where $\boldsymbol{z} \in \mathbb{C}^d$
F_{z_j}	Partial derivative of $F : \mathbb{C}^d \to \mathbb{C}$ with respect to the variable z_j
$\mathcal{V}$	The singular variety of the function $F(\boldsymbol{z})$
$T_{\boldsymbol{a}}(\boldsymbol{r})$	$\{\boldsymbol{z} \in \mathbb{C}^d : \lvert z_i - a_i\rvert \leq \lvert r_i\rvert, i \in [d]\}$, the polytorus centered at the point $\boldsymbol{a} \in \mathbb{C}^d$ of polyradius $\boldsymbol{r} \in \mathbb{C}^d$. By $T(\boldsymbol{r})$ we mean the polytorus of polyradius $\boldsymbol{r}$ centered at the origin.
$D_{\boldsymbol{a}}(\boldsymbol{r})$	$\{\boldsymbol{z} \in \mathbb{C}^d : \lvert z_i - a_i\rvert \leq \lvert r_i\rvert, i \in [d]\}$ the polydisk centered at the point $\boldsymbol{a} \in \mathbb{C}^d$ of polyradius $\boldsymbol{r} \in \mathbb{C}^d$. By $D(\boldsymbol{r})$ we mean the polydisk of polyradius $\boldsymbol{r}$ centered at the origin.
diag(F)	The *diagonal* of a function $F : \mathbb{C}^d \to \mathbb{C}$ having power series $F = \sum_{\boldsymbol{i} \in \mathbb{N}^d} f_{\boldsymbol{i}} \boldsymbol{z}^{\boldsymbol{i}}$, given by $diag(F)(z) := \sum_{n=0}^{\infty} f_{n,...,n} z^n$.

CHAPTER 1

INTRODUCTION

1.1 Motivation and History

The problem of deciding whether a given multivariate rational function, analytic at the origin, has positive power series coefficients was originally motivated by H. Lewy and K. Friedrichs, who, in 1930, were studying finite difference approximations of the wave equation. In the course of their work, they needed to prove that the following rational function

$$\frac{1}{(1-x)(1-y)+(1-x)(1-z)+(1-y)(1-z)} = \sum_{k,m,n\geq 0} a(k,m,n)x^k y^m z^n$$

had Taylor series coefficients $a(k,m,n)$ which were all positive. Not knowing how to proceed, they sent their question to G. Szegő who, using tools from the theory of special functions, was able to solve the problem [2]. In 1933, Kaluza gave another proof of the postivity of Lewy and Friedrichs' function [3], this time using only elementary techniques. From there, the problem became ingrained in the mathematical consciousness, with authors periodically presenting results proving the positivity of functions, either towards applications or eventually because the problem had become interesting in its own right. For more history on the positivity problem as well as a survey of results, see [4].

The techniques used in "classical" positivity proofs vary widely: special functions, integral methods and transforms, determinantal polynomials, combinatorial identities, positivity-preserving operators, and computer algebra have all been employed to one degree or another in proving positivity of various functions.

This is to be contrasted with the "ACSV"-centric approach, which relies on the theory of *analytic combinatorics in several variables*, pioneered by R. Pemantle and M. Wilson in the early 2000s and still in active development to this day. (See: `http://acsvproject.com`) This powerful and beautiful theory, which utilizes parts of several complex variables, algebraic geometry, topology and Morse theory to determine the asymptotic behavior of multivariate sequences, is a natural generalization of the well-studied univariate theory of analytic combinatorics, and can be applied to a variety of discrete structures and phenomena in mathematics and the natural sciences, including the positivity problem.

With the development of ACSV came the potential for proving *eventual* positivity of a rational function's power series coefficients– at least in a certain direction. A. Straub investigated when simpler conditions –such as positivity along the main diagonal– were sufficient for showing complete positivity of the function [4]. Using ACSV techniques, mathematicians have found ways to prove that many such rational functions are asymptotically positive. That is, for any direction in $\mathbb{Z}^d$, the coefficients of our generating function which lie along that direction are eventually positive. This is done by treating our rational function as a generating function for a sequence, applying the ACSV theory to derive an asymptotic expansion for it in one or more crucial directions (for instance, the main diagonal), then using results from classical positivity to extrapolate this to all coefficients. For one example of the use of ACSV in proving asymptotic positivity of several functions' main diagonal, see Baryshnikov et al.'s paper [5].

One other key aspect of ACSV theory that cannot be overstated is its predisposition for *effective computation*. In theory, a *vast* majority of its results are explicit enough to implement in computer algebra systems. A considerable amount of effort has gone in this direction, with articles and book chapters devoted to the possibility of effective asymptotics for certain classes of functions, including implementation in various computer algebra systems like SAGE, Maple, and Python. [6] [7] [8] [9]

This tendency towards effectivity blends particularly well with our work, which would theoretically allow one to prove complete positivity of our power series coefficients in a certain direction *given* asymptotic positivity in that direction. We describe this procedure in the next section.

1.2 Complete positivity via asymptotic positivity

This thesis attempts to address the following question:

> "Given a rational function that we *know* is asymptotically positive in a direction, can we provide a bound for the number of terms we would need to check for positivity before the leading term of our asymptotic dominates all error terms, hence positivity of higher-indexed coefficients follows from positivity of our asymptotic?"

If this were the case, then complete positivity along a given direction could be decided by simply:

1. Using (effective) procedures from ACSV to derive an asymptotic along this direction, verifying that it is eventually positive.

2. Computing an index at which the leading term of the asymptotic dominates all error terms, implying all higher-index coefficients are positive.

3. Using a computer to check the first finitely many terms up to this index, verifying they're all positive.

This thesis attempts to take the first steps towards the construction of a general procedure for completing Step 2 of the above. Here, we carry out Steps 1-3 above in full detail on a couple of specific examples of interest, as well as make observations regarding the feasibility of checking the first finitely many coefficients for positivity using current computer technology.

We should note that the question motivating Step 2 is quite a natural one to ask. Indeed, it had already been considered by Dong, Melczer, and Mezzarobba in [10] with respect to the class of P-recursive sequences, which contains the diagonals of the functions we consider by a result of [11]. In [10], the authors provide a practical algorithm for computing the explicit error bounds associated with the singularity analysis of such sequences, which is one of two major components of the process described here.

In the next section we provide the necessary ACSV background, as well as detail our "Main Algorithm" for the determination of complete positivity in a direction as applied to our examples.

CHAPTER 2

BACKGROUND

Here, we introduce the key ideas and notations from ACSV used in our calculations, as well as a summary of our approach in computing our desired index at which the asymptotic expansion for our coefficients "takes over" in terms of forcing positivity of all higher terms. We shall call this index the *final index* of the sequence, and denote it by N_f throughout.

Finally, we shall give a bit of motivation and background regarding the families of functions to which we apply our techniques in the later sections.

2.1 ACSV

For a thorough treatise on ACSV aimed at researchers in the field, see [12]. Our terminology and notation mirrors that of [13]; we refer the reader to §3.1, 3.2 and 5.2 to become familiar with the definitions we use.

Suppose we're given a rational function in d complex variables:

$$F(\boldsymbol{z}) = \frac{G(\boldsymbol{z})}{H(\boldsymbol{z})}$$

with the property that the polynomial $H(\boldsymbol{z})$ is nonzero at the origin. Then, completely analogous to the theory for a single complex variable, we have a power series representation for F which converges absolutely on some *polydisk* $D_{\mathbf{0}}(\boldsymbol{a})$ centered at $\mathbf{0}$, i.e. F is *analytic* at the origin with series:.

$$F(\boldsymbol{z}) = \sum_{\boldsymbol{j}\in\mathbb{N}^d} f_{\boldsymbol{j}} \boldsymbol{z}^{\boldsymbol{j}},$$

where the series coefficients $f_{\boldsymbol{j}} = f_{(r_1,r_2,\ldots,r_d)}$ are given by the Cauchy integral formula (suitably generalized):

$$f_{\boldsymbol{j}} = \frac{1}{(2\pi i)^d}\int_{T(\boldsymbol{b})} \frac{F(\boldsymbol{z})}{\boldsymbol{z}^{\boldsymbol{j}+\mathbf{1}}} d\boldsymbol{z}$$

for all $\boldsymbol{j}$ in $\mathbb{N}^d$, $\boldsymbol{b}$ being some point in the disk $D_0(\boldsymbol{a})$.

Fix a direction vector $\boldsymbol{r} \in \mathbb{N}^d$. We wish to derive an asymptotic expansion for the sequence $(f_{n\boldsymbol{r}})_{n\in\mathbb{N}}$. As is the case for analytic combinatorics of a single variable, the general workflow for performing an asymptotic analysis of such a sequence is the following:

1. Bound $\rho = \limsup_{n\to\infty} |f_{n\boldsymbol{r}}|^{1/n}$, the exponential growth of the coeffi-

cients. This is the dominant growth rate of our sequence.

2. Determine the singularities of F that contribute (the most) to our asymptotic behavior.

3. Write down the Cauchy integral for our sequence; "localize" it and compute residues.

4. Apply the Saddle-point method to determine the asymptotic.

Our work utilizes only the *smooth* theory of ACSV, that is, the case when the singular variety $\mathcal{V}$ of the function F consists of points such that the squarefree part of H has at least one nonvanishing partial derivative. Baryshnikov, Pemantle, Wilson, and others have developed ACSV to include the non-smooth case; it is currently in the process of being made effective. (See the latter chapters of Melczer's book [9]). Someday we would like to extend our methods to include such functions. See Section 5.2.

2.2 Computing N_f

Here, we describe at a high level our procedure [1] and rationale in calculating an index at which our asymptotic dominates:

INPUT:

- a rational function $F = G/H$ in d variables

- a direction vector $\boldsymbol{r} \in \mathbb{N}^d$. For this paper, we always take $\boldsymbol{r} = \mathbf{1} := (1, 1, ..., 1)$.

ASSUMING:

- $H(\mathbf{0}) \neq 0$

- $\mathcal{V} = \mathcal{V}(H)$ *smooth*

[1]Strictly speaking, what we describe is not a *procedure* in the proper sense of the word; we do not know in advance whether we are able to accomplish certain steps. Indeed, much of Steps 1 and 2 depend on external theory, which can be comprehensive or very limited depending on the function at hand. Our 'procedure' is really more of a recipe or *set of guiding principles* with the potential to be made effective for certain classes of functions. Henceforward when we refer to our "procedure" / "Main Algorithm" we simply mean this set of guiding principles, unless we specify otherwise.

- F admits a *nondegenrate strictly minimal smooth contributing point* $\boldsymbol{w} \in \mathbb{C}_*^d$.
- The coefficinents $f_{n\boldsymbol{r}}$ are eventually positive.

OUTPUT: An $N_f \in \mathbb{N}$ such that $\forall n \geq N_f$, $f_{n\boldsymbol{r}} > 0$.
PROCEDURE:

1. Verify that our set of assumptions hold for our inputs.
2. Run the smooth ACSV procedure on our inputs, obtaining:
   - a complex multivariate saddle point integral χ as described in Chapter 5 of [13], which can be viewed as $\chi(n)$ depending on the parameter n.
   - a function $\lambda(n)$ such that $\chi(n)$ (hence $f_{n\boldsymbol{r}}$) is asymptotically equivalent to $\lambda(n)$.
3. Compute numbers $\tau \in (0, |\boldsymbol{w}^{-\boldsymbol{r}}|)$, $c > 0$ such that $\forall n \in \mathbb{Z}_+, |f_{n\boldsymbol{r}} - \lambda| < c\tau^n$.
4. Show that there exists an $L \in \mathbb{Z}_+$ such that $\forall n \geq L$, $|\chi - \lambda| \leq \lambda - c\tau^n$.
5. This $L = N_f$. Output it.

DONE.

Note that the inequality in Step 3 is strict. Also, our assumptions on our inputs are enough to guarantee the existence of the τ, c, and N_f; the existence of τ, c is due to the contents of Lemma 5.1, [14], while N_f's follows from the fact that $|\boldsymbol{w}^{-\boldsymbol{r}}|$, which bounds the growth in λ, is a tight bound on the growth of χ.

Our L from Step 4 *is* a valid N_f (Step 5) by a basic application of the triangle inequality, the positivity of λ, and the fact that $\lambda \sim f_{n\boldsymbol{r}}$ i.e. $\lim_{n\to\infty} \frac{\lambda(n)}{f_{n\boldsymbol{r}}} = 1$.

2.3 Background for Bivariate, Gillis-Reznick-Zeilberger function classes

2.3.1 Bivariate case

We consider functions $F : \mathbb{C}^2 \to \mathbb{C}$ of the form

$$F(x,y) = \frac{1}{1 - ax - by + cxy}, \quad a, b \geq c > 1.$$

Our primary motivation for this class of functions is its simplicity. Indeed, not only are 2-variable functions inherently easier to compute with by hand, but some of the problems associated with running our procedure on functions in many variables can be sidestepped. For example, notice that the optimization problem on which we find ourselves stuck in Step 3 of the GRZ problem (See Section 4.3, Conjecture 1) completely disappears in 2 variables, as the corresponding step in this case is a routine 1D optimization problem.

However, even bivariate functions such as ours can be of academic interest outside of being used as 'toy examples.' For instance, [7] describes a different method for effective computation of asymptotics for a larger class of bivariate functions. [15] gives several examples of combinatorial problems whose 2D sequences have generating functions belonging to our class. [4] discusses necessary and sufficient conditions for positivity of a class of bivariate functions intersecting ours.

Note that, initially, we thought about letting a, b, c vary over the range defined by the inequalities $a, b \geq c > 0$. However, this introduces a non-smooth point in our singular variety $\mathcal{V}$ for the function F; we are forced to adopt the case of the hyperplane arrangement (see Chapter 8 of [9]), with which we are not as familiar. We hope to extend to include this case in the future. See Section 5.

2.3.2 Gillis-Reznick-Zeilberger (GRZ) case

Fix an integer $d \geq 4$. We consider the function $F_{d!,d} : \mathbb{C}^d \to \mathbb{C}$ defined by

$$F_{d!,d}(\boldsymbol{z}) = \frac{1}{1 - z_1 \cdots - z_d + d! z_1 \cdot \cdots \cdot z_d}, \quad \forall \boldsymbol{z} \in \mathbb{C}^d.$$

The positivity of this function, as alluded to earlier, was originally studied in [1]. Actually, the authors of that paper considered a wider class of functions of the form

$$F_{c,d}(x_1, ..., x_d) = \frac{1}{1 - e_1 + ce_d},$$

where c is real and e_i is the ith elementary symmetric function in the variables $x_1, \ldots, x_d$. By Prop. 5.5 of [14] the case when $c < 0$ is trivial, so it suffices to consider the case when $c > 0$. Through a combination of their results and [16], one can conclude that $F_{c,3}$ has nonnegative coefficients iff $c \leq 4$. GRZ also conjecture that for $d \geq 4$, $F_{c,d}$ has nonnegative coefficients iff $c \leq d!$, and they prove that nonnegativity of $diag(F_{d!,d})$ implies nonnegativity of $F_{d!,d}$ in this case.

Kauers [17] and Pillwein [18] together proved that the diagonal coefficients were indeed positive for $d = 4, \ldots, 17$, using cylindrical algebraic decomposition.

More recently, Yu answered the problem in general for $d \geq 4$ [19] by looking at the roots of certain polynomials and power series. Our result re-proves it using ACSV, demonstrating the potential for (effective) ACSV to apply to novel positivity problems.

CHAPTER 3

STATEMENT OF RESULTS

Here we state our findings regarding our bounds. We also state some lemmas needed during the course of our computations; their proofs are in the next section.

3.1 General use lemmas

The following results we found useful for both our bivariate and GRZ examples; we believe they would be of use in *any* similar analysis of rational functions, hence, this is why they've been designated their own separate subsection.

Lemma 1. *(Calculation involving the Product Log):*
Let $\mathcal{H}$ be a positive real, α a rational strictly between $1/3$ and $1/2$. There exists some $\tilde{N} \in \mathbb{Z}_+$, so that $\forall n \geq \tilde{N}$, $e^{-\frac{\mathcal{H}}{2}n^{1-2\alpha}} < n^{\frac{1}{2}-3\alpha}$, namely

$$\tilde{N} := 1 + \left\lceil \max\left\{ \left(\frac{2}{\mathcal{H}e}\right)^{1/(\frac{3}{2}-5\alpha)}, \left[\frac{6\alpha-1}{2\alpha-1}W_0\left(\frac{2\alpha-1}{6\alpha-1}\mathcal{H}\right)\right]^{1/(1-2\alpha)}, \left[\frac{6\alpha-1}{2\alpha-1}W_{-1}\left(\frac{2\alpha-1}{6\alpha-1}\mathcal{H}\right)\right]^{1/(1-2\alpha)} \right\} \right\rceil,$$

where W_0, W_{-1} are the corresponding branches of the Lambert W-function / product log, defined in Section 4.1.

Lemma 2. *(Comparing $c\tau^n$ and $\lambda(n)$):*
Let $c, D > 0$, $\boldsymbol{w} = (w_1, \ldots w_d) \in (\mathbb{C} - \mathbf{0})^d$ and $0 < \tau < |w_1|^{-1} \cdot \ldots \cdot |w_d|^{-1}$. Then,

$$\exists N \in \mathbb{Z}_+ \text{ such that } \forall n \geq N, c\tau^n < D|w_1 \cdot \ldots \cdot w_d|^{-n} n^{(1-d)/2}. \qquad (*)$$

Moreover, we have an algorithm / formulas for this N, which we describe in the proof of the lemma.

3.2 Bivariate Case

Computation 1. *(Bound for family of bivariate functions):*
Let $F : \mathbb{C}^2 \to \mathbb{C}$ be defined by

$$F(x,y) = \frac{1}{1 - ax - by + cxy}, \quad a, b \geq c > 1.$$

Put

$$\boldsymbol{w} = (w_1, w_2) := \frac{ab - \sqrt{(ab)^2 - abc}}{ac}\left(1, \frac{a}{b}\right).$$

Then, to prove positivity of the diagonal coefficients of F's power series centered at the origin, it suffices to check the first N_f terms for positivity. N_f is given by the formula:

$$N_f = \max\left\{N_2, \lceil \delta^{\frac{-1}{\alpha}} \rceil, \left\lceil \left(\frac{c_0}{\mu}\right)^{1/(3\alpha-1)} \right\rceil, \tilde{N}, \left\lceil \left(\frac{c_{16}}{\epsilon}\right)^{1/(3\alpha-1)} \right\rceil, N\right\}$$

where $0 < \delta < \min\left\{w_2, \pi/2, \frac{1}{2}\ln(b/(cw_1))\right\}$, $0 < \epsilon < 1$, $\mu > 0$, *and* $\alpha \in \mathbb{Q} \cap (1/3, 1/2)$ *are freely chosen parameters, and the constants* $N_2, c_0, \tilde{N}, c_{16}$, *and* N *are obtained effectively in the derivation described in the next section.*

In the special case when $a = b$, we also have the following corollary, whose proof is immediate by Theorem 2.3 of [4]:

Corollary 1. *Let F be as described in Computation 1, with* $a = b > c > 1$. *Then, to prove positivity of* all *of F's power series coefficients, in* any *direction, it suffices to check the first N_f terms along the diagonal.*

3.3 GRZ Case

As for the GRZ class, we have:

Computation 2. *(Bound for GRZ function):*
Let $d \geq 4$ be an integer. Let $F_{d!,d} : \mathbb{C}^d \to \mathbb{C}$ be defined by

$$F_{d!,d}(\boldsymbol{z}) = \frac{1}{1 - z_1 \cdots - z_d + d! z_1 \cdot \cdots \cdot z_d}, \quad \forall \boldsymbol{z} \in \mathbb{C}^d.$$

Then, to prove positivity of the diagonal coefficients of $F_{d!,d}$'s power series centered at the origin, it suffices to check the first N_f terms for positivity. N_f is given by the formula:

$$N_f = \max\left\{N_2, N_{16}, \tilde{N}, \left\lceil \left(\frac{c_{18}}{\epsilon}\right)^{1/(3\alpha-1)} \right\rceil, N\right\}$$

with the freely chosen parameters:

- $\delta \in (0, \frac{1}{(d-1)^{d-1}})$, *chosen so that* $\frac{1}{\rho \cdot (d-1)} - 1 \geq |e^{2\delta - 1}|$, *($\rho$ being the unique real root of a polynomial described in Section 4),*

- $0 < \epsilon, \tilde{\epsilon} < 1$

- $\alpha \in (1/3, 1/2) \cap \mathbb{Q}$

- $\mu > 0$,

along with the constants N_{16}, c_{18}, N, and $\tilde{N}$ which are computed as in Section 4.3, and N_2 being a Big-O constant associated with a certain estimate as described in 4.3.

Note that the result holds trivially for $d = 1$, but is false for $d = 2, 3$. [4] Using Proposition 3 in their paper [1], the result of Computation 2 allows us to answer *affirmatively* the original conjecture of Gillis, Reznick, and Zeilberger.

CHAPTER 4

PROOFS / DERIVATIONS

4.1 Proofs of General-Use Lemmas

We use the following well-known special function in our results:

Definition 1. *(Product log / Lambert-w function):*
The Lambert w-function *or* product log *is the (set-valued) inverse function of $f(w) = we^w$, where w is a complex variable. More specifically, for each integer k we have a branch $W_k(z)$ of the Lambert-w function, W_k being a complex-valued function of a complex variable. Taken together, these W_k have the property that for any complex numbers z and w, $we^w = z$ holds iff $w = W_k(z)$ for zome $k \in \mathbb{Z}$. When dealing with real numbers, it suffices to consider W_0 and W_{-1} only, for reasons mentioned in [20]*

4.1.1 Proof of Lemma 1

Proof. By an elementary limit calculation, we have $e^{-\frac{\mathcal{H}}{2}n^{1-2\alpha}} = o(n^{\frac{1}{2}-3\alpha})$, implying that $e^{-\frac{\mathcal{H}}{2}n^{1-2\alpha}} < n^{\frac{1}{2}-3\alpha}$ eventually. Thus to prove the lemma it suffices to find the largest real solution of the equation

$$e^{-\frac{\mathcal{H}}{2}n^{1-2\alpha}} = n^{\frac{1}{2}-3\alpha}, \qquad (*)$$

if one exists. But $n \in \mathbb{R}$ solves $(*)$ iff it solves $(-\frac{\mathcal{H}}{2}n^{1-2\alpha})e^{-\frac{\mathcal{H}}{2}n^{1-2\alpha}} = -\frac{\mathcal{H}}{2}n^{3/2-5\alpha}$, which holds iff n is such that $-\frac{\mathcal{H}}{2}n^{1-2\alpha} = W_k(-\frac{\mathcal{H}}{2}n^{3/2-5\alpha})$, for some $k \in \mathbb{Z}$. [20] Since n is real, it suffices to only consider $k \in \{0, -1\}$; the above equation $(*)$ can be solved for $-\frac{\mathcal{H}}{2}n^{1-2\alpha}$ precisely when $-\frac{\mathcal{H}}{2}n^{3/2-5\alpha} \geq \frac{-1}{e}$, which holds only when $\left(\frac{2}{\mathcal{H}e}\right)^{1/(\frac{3}{2}-5\alpha)} \leq n$. Thus, in particular, any real n solving (*) must be positive. Under this additional assumption and our hypothesis on $\mathcal{H}$ and α, we can rewrite our equation as $n^{3\alpha} = \sqrt{n}e^{\mathcal{H}/2*n(1-2\alpha)}$. Some simple algebra then yields our only two valid real solutions to (*),

$$n = \left[\frac{6\alpha-1}{2\alpha-1}W_0(\frac{2\alpha-1}{6\alpha-1}\mathcal{H})\right]^{1/(1-2\alpha)}, \left[\frac{6\alpha-1}{2\alpha-1}W_{-1}(\frac{2\alpha-1}{6\alpha-1}\mathcal{H})\right]^{1/(1-2\alpha)}.$$

Thus, we certainly have a largest real solution, if one exists. If no real solution exists, then $\tilde{N} = 1$ works by the intermediate value theorem. Combining cases, we see that the $\tilde{N}$ described in the statement will work. □

4.1.2 Proof of Lemma 2

Proof. Note that $(*)$ is equivalent to the statement

$$\exists N \in \mathbb{Z}_+ \text{ such that } \forall n \geq N, \frac{c}{D} n^{(d-1)/2} < \left(\frac{|w_1 \cdot \ldots \cdot w_d|^{-1}}{\tau} \right)^n,$$

and the existence of such an N follows from a result of elementary calculus. So it suffices to find the largest real n which solves $\frac{c}{D} n^{(d-1)/2} = \left(\frac{|w_1 \cdot \ldots \cdot w_d|^{-1}}{\tau} \right)^n$. Calling $A := \frac{|w_1 \cdot \ldots \cdot w_d|^{-1}}{\tau}$ and $b := (d-1)/2$, this is equivalent to

$$[(c/D)^{\frac{1}{b}} n]^b = e^{n \log A}. \qquad (+)$$

Now we split into 4 cases:

1. $\mathbf{d \equiv 1 (mod 4)}$: Then $b \in \mathbb{Z}$ is even, thus $(+)$ holds iff

$$-n e^{-n \frac{\log A}{b}} = \mp \left(\frac{c}{D} \right)^{-1/b} \qquad (++)$$

holds. Since $\left(\frac{c}{D} \right)^{-1/b}$ is positive, the positive branch of the equation $(++)$ will *certainly* have a real solution (the standard one– see [20]) in terms of W_0, the principal branch of the Lambert W-function. Specifically:

$$n = -W_0 \left(\left(\frac{c}{D} \right)^{-1/b} \right).$$

If, in addition, $\left(\frac{c}{D} \right)^{-1/b} \leq \frac{1}{e}$, then we'll have two additional solutions for the negative branch of $(++)$, one involving W_0 and the other involving W_{-1}:

$$n = -W_0 \left(- \left(\frac{c}{D} \right)^{-1/b} \right) \quad \text{and} \quad n = -W_{-1} \left(- \left(\frac{c}{D} \right)^{-1/b} \right).$$

Regardless, for each branch (positive or negative) of $(++)$ having solutions in the above form, we can solve for $n \in \mathbb{R}$ uniquely and note that any larger integer will work as an "N_i" for that branch. I.e., put

$$N_1 := \left\lceil -W_0 \left(- \left(\frac{c}{D} \right)^{-1/b} \right) \right\rceil,$$

$$N_2 := \left\lceil \max_{k=0,-1} \left\{ -W_k \left(-\left(\frac{c}{D}\right)^{-1/b} \right) \right\} \right\rceil.$$

To obtain our desired N, simply maximize over whichever elements of $\{N_1, N_2\}$ come from a branch of (++) having a solution.

2. $\mathbf{d \equiv 3(mod4)}$: Then b is an odd integer, so the LHS of (+) has a unique positive bth root, giving us

$$(c/D)^{\frac{1}{b}} n = e^{n(\log A)/b}.$$

Through the same sort of manipulations as in Case 1, we obtain the same equation as (++), except now we only have the *negative* branch to worry about. If this equation has no solution, then we have no real solutions of $(*)$, in which case we can simply try larger and larger positive integer values of n until we find one for which $RHS(*) > LHS(*)$. (This is guaranteed to happen, since the RHS is eventually larger.) Then, by IVT, this will be our N. If, on the other hand, $(*)$ *does* have a real solution, then proceed as in Case 1 to determine N depending on whether $-\left(\frac{c}{D}\right)^{-1/b}$ is positive or in $[-1/e, 0)$.

3. $\mathbf{d \equiv 0, 2(mod4)}$: Then b is not an integer, and by analogous work we obtain the same equation as (++), except this time we can only have a solution in the *positive* branch of this equation. The rest is exactly the same as in Case 2.

Note that the "algorithm" outlined above is guaranteed to terminate, as the only place it could possibly fail is in Cases 2 and 3 when we have no solution and so try larger and larger values of n. But, as we discussed, we are guaranteed to eventually reach an n which *will* work. □

4.2 Derivation of Bivariate Bound

Proof. We start with

$$F(x,y) = \frac{1}{1 - ax - by + cxy}.$$

Let us denote the denominator of F by $H(x,y) := 1 - ax - by + cxy$, and observe that the denominator is squarefree. As a first step, we need to verify that the hypotheses of the smooth theory apply. This means determining whether F admits a strictly minimal nondegenerate smooth contributing point in the direction of our diagonal, $\mathbf{1}$, and is such that $H_y \neq 0$ at this point.

Towards this end, the singular variety for F can easily be computed by finding all zeros of our denominator function, giving:

$$\mathcal{V} = \{(x,y) \in \mathbb{C}^2 : x \neq \frac{b}{c} \wedge y = \frac{ax-1}{cx-b}\}.$$

$\mathcal{V}$ is also seen to be smooth, as $H_x = -a + cy$, $H_y = -b + cx$, so if $(x,y) \in \mathcal{V}$ then $H_y \neq 0$.

Next, compute the set of critical points along the main diagonal, which we denote $\mathrm{crit}(\mathbf{1})$. This is done by solving the smooth critical point equations for $\boldsymbol{w} \in \mathbb{C}_*^d$, for general d:

$$\begin{cases} H(\boldsymbol{w}) = 0 \\ w_1 H_{z_1}(\boldsymbol{w}) - w_j H_{z_j}(\boldsymbol{w}) = 0, \quad 2 \leq j \leq d \end{cases}$$

with the additional requirement that for some $j \in [d]$, $H_{z_j}(\boldsymbol{w}) \neq 0$. For simple rational functions, this step can usually be accomplished automatically, and it certainly can be in the case at hand. We find

$$\mathrm{crit}(\mathbf{1}) = \left\{ \frac{ab \pm \sqrt{(ab)^2 - abc}}{ac} \left(1, \frac{a}{b}\right) \right\}.$$

To determine the *minimal* points lying in $\mathrm{crit}(\mathbf{1})$, it suffices by Proposition 5.4 of [14] to find $\boldsymbol{w} \in \mathrm{crit}(\mathbf{1})$ such that for no $\boldsymbol{v} \in \mathcal{V}$ is $|\boldsymbol{v}| = t|\boldsymbol{w}|$, for some $t \in (0,1)$.

Through assuming a contradiction and then performing a simple computation involving comparison of magnitudes (which can also be accomplished algorithmically), we see that

$$\boldsymbol{w} = (w_1, w_2) := \frac{ab - \sqrt{(ab)^2 - abc}}{ac} \left(1, \frac{a}{b}\right)$$

is minimal. We claim that this minimality is *strict*, i.e. that $T(\boldsymbol{w}) \cap \mathcal{V} = \{\boldsymbol{w}\}$.

To see this, one can solve the 2×2 system derived from $H(w_1 e^{it_1}, w_2 e^{it_2}) = 0$ for $t_1, t_2 \in [0, 2\pi)$. The only solution is found at $(0,0)$, thus since we are working with a power series and our direction vector has only positive coordinates, Proposition 3.6 of [13] implies $\boldsymbol{w}$ is strictly minimal. Finally, to verify that $\boldsymbol{w}$ is quadratically *nondegenerate*, one can compute directly the matrices involved in Lemma 5.5 of [14]. From this, we see that $\boldsymbol{w}$ is indeed nondegenerate, thus the smooth theory applies, and by established results such as Theorem 5.2 of [14] we obtain an "out-of-the-box" asymptotic expansion for the diagonal coefficients $f_{n\mathbf{1}}$ of our series:

$$f_{n\mathbf{1}} = \frac{(w_1 w_2)^{-n}}{\sqrt{2\pi n}} \cdot \frac{1}{\sqrt{\det(\mathcal{H})} w_2 (b - c w_1)} (1 + O(1/n)),$$

where $\mathcal{H}$ is the matrix obtained in the computation from Lemma 5.5. (It is the $d-1 \times d-1$ Hessian matrix of the function ϕ at the origin, which will be computed later.)

Call the leading term in this expansion λ:

$$\lambda(n) := \frac{(w_1 w_2)^{-n}}{\sqrt{2\pi n}} \cdot \frac{1}{\sqrt{\det(\mathcal{H})} w_2 (b - c w_1)}.$$

It is clear that λ is positive, and from the fact that $f_{n\mathbf{1}} \sim \lambda$ it is immediate that $f_{n\mathbf{1}}$ is eventually positive.

Numerics suggest that the early terms in our diagonal sequence are all positive as well. Thus this is a good candidate for our process of proving total positivity by 'finding N_f,' which we shall do now. Note that at this point, we have completed Step 1 and part of Step 2 of the procedure described in Section 2.2. To find χ (completing Step 2) and then complete Step 3, we first let $\delta > 0$ be any value less than

$$\min \left\{ w_2, \pi/2, \frac{1}{2} \ln(b/(c w_1)) \right\}.$$

As usual, we introduce the apparatus associated with any multivariate asymptotic analysis. Put $\mathcal{T} = T(\hat{\boldsymbol{w}}) = \{z \in \mathbb{C} : |z| = w_1\}$, and define a subset $\mathcal{N} \subset \mathcal{T}$ by:

$$\mathcal{N} = \{z \in \mathcal{T} : Arg(z) \in (-\delta, \delta)\},$$

which can be parametrized as

$$\mathcal{N} = \{w_1 e^{i\theta_1} : \theta_1 \in (-\delta, \delta)\}$$

by mapping $z \to w_1 e^{iArg(z)}$. Note that this set is open in the topology of the torus. We also let $g : \mathcal{T} \to \mathbb{C}$ be defined by $g(x) := \frac{ax-1}{cx-b}$. It is simple to check that the map g is holomorphic (hence analytic) on $\mathcal{T}$. Next we need to show that this choice of $\delta, \mathcal{N}, g$ satisfy requirements (i) - (iii) given on page 206 of [14]. This will allow us to deform our Cauchy integral contour so that it is "close to" our contributing point, thereby letting us perform a local singularity analysis.

The verifications of (i) and (iii) are simple. For (ii), for our purposes we will actually need to prove something slightly stronger. Putting $\mathcal{N}' := \mathcal{T} - \mathcal{N}$, we need to show (constructively) that:

1. $\exists \eta \in \mathbb{R}_+$ such that $\forall \hat{\boldsymbol{z}} \in \mathcal{N}, w_2 - \delta < w_2 \leq |g(\hat{\boldsymbol{z}})| \leq \eta < w_2 + \delta$, and
2. $\exists \zeta \in \mathbb{R}_+$ such that $\forall \hat{\boldsymbol{z}} \in \mathcal{N}', w_2 - \delta < w_2 < \zeta \leq |g(\hat{\boldsymbol{z}})|$.

To prove 1., first parametrize $|g(\hat{\boldsymbol{z}})|$ on $\bar{\mathcal{T}}$ by:

$$h : [-\pi, \pi] \to \mathbb{R}_+, \quad h(\theta_1) := |g(w_1 e^{i\theta_1})|, \ \forall \theta_1 \in [-\pi, \pi].$$

Taking the norm,

$$h(\theta_1) = \sqrt{\frac{(aw_1 cos\theta_1 - 1)^2 + (aw_1 sin\theta_1)^2}{(cw_1 cos\theta_1 - b)^2 + (cw_1 sin\theta_1)^2}},$$

and because $\mathcal{T}$ is a compact subset of $\mathbb{C}$ and g is continuous on $\mathcal{N}$ (because it is holomorphic), h must obtain a max and min on $[-\delta, \delta]$. Also, because our change of variables from $(-\delta, \delta)$ is differentiable, g holomorphic on *all* of $\mathcal{T}$, it follows that h will be differentiable and everywhere positive on $[-\pi, \pi]$. Finally, notice that h is even.

With these facts in mind, a straightforward (if not slightly tedious) Calc-I-style computation shows that:

- Our minimum on $\mathcal{T}$ will be w_2, occurring at $\theta_1 = 0$
- Our maximum on $\mathcal{N}$ will be on the boundary of $\mathcal{N}$, i.e. at $\pm\delta$

- Our minimum on $\mathcal{N}'$ will occur at the same point, and

- Our maximum on $\mathcal{N}'$ will occur at $\pm\pi$.

Now, observe that

$$w_2^2 < h(\delta)^2 = \frac{(aw_1 cos\delta - 1)^2 + (aw_1 sin\delta)^2}{(cw_1 cos\delta - b)^2 + (cw_1 sin\delta)^2}, \quad \delta \in (0, \pi)$$

..seen, for instance, via Reduce. So we can take $\zeta, \eta := h(\delta)$ for the delta given above. Through a similar computation one sees that $h(\delta) < w_2 + \delta$, thus completing the proof of (ii).

In general, a point whose first $d-1$ coordinates are in $\mathcal{T}$ and whose last coordinate has magnitude $z_d - \delta$ will be in the domain of convergence for our power series for F, hence by the Cauchy integral formula, our diagonal coefficients will be expressible as a Cauchy integral I. We introduce I, along with its associated "localized" integrals from which we will derive saddle point estimates. For general d, F, and strictly minimal smooth contributing pint $\boldsymbol{w}$ of F these integrals would be:

$$f_{n\mathbf{1}} = I := \frac{1}{(2\pi i)^d} \int_{\mathcal{T}} \left(\int_{|z_d| = |w_d| - \delta} F(\boldsymbol{z}) \frac{dz_d}{z_d^{n+1}} \right) \frac{d\hat{\boldsymbol{z}}}{\hat{\boldsymbol{z}}^{(n+1)\hat{\mathbf{1}}}},$$

$$I_{\text{loc}} := \frac{1}{(2\pi i)^d} \int_{\mathcal{N}} \left(\int_{|z_d| = |w_d| - \delta} F(\boldsymbol{z}) \frac{dz_d}{z_d^{n+1}} \right) \frac{d\hat{\boldsymbol{z}}}{\hat{\boldsymbol{z}}^{(n+1)\hat{\mathbf{1}}}},$$

$$I_{\text{out}} := \frac{1}{(2\pi i)^d} \int_{\mathcal{N}} \left(\int_{|z_d| = |w_d| + \delta} F(\boldsymbol{z}) \frac{dz_d}{z_d^{n+1}} \right) \frac{d\hat{\boldsymbol{z}}}{\hat{\boldsymbol{z}}^{(n+1)\hat{\mathbf{1}}}}, \text{ and}$$

$$\chi := I_{\text{loc}} - I_{\text{out}} = \frac{-1}{(2\pi i)^d} \int_{\mathcal{N}} \left(\int_{|z_d| = |w_d| + \delta} F(\boldsymbol{z}) \frac{dz_d}{z_d^{n+1}} - \int_{|z_d| = |w_d| - \delta} F(\boldsymbol{z}) \frac{dz_d}{z_d^{n+1}} \right) \frac{d\hat{\boldsymbol{z}}}{\hat{\boldsymbol{z}}^{(n+1)\hat{\mathbf{1}}}},$$

and in our specific example these reduce to:

$$I = \frac{1}{(2\pi i)^2} \int_{|x| = w_1} \int_{|y| = w_2 - \delta} F(x, y) \frac{dy dx}{y^{n+1} x^{n+1}},$$

$$I_{loc} = \frac{1}{(2\pi i)^2} \int_{\mathcal{N}} \int_{|y| = w_2 - \delta} F(x, y) \frac{dy dx}{y^{n+1} x^{n+1}},$$

$$I_{out} = \frac{1}{(2\pi i)^2} \int_{\mathcal{N}} \int_{|y|=w_2+\delta} F(x,y) \frac{dydx}{y^{n+1}x^{n+1}},$$

$$\chi := I_{\rm loc} - I_{\rm out} = \frac{-1}{(2\pi i)^2} \int_{\mathcal{N}} \left(\int_{|y|=w_2+\delta} F(x,y)\frac{dy}{y^{n+1}} - \int_{|y|=w_2-\delta} F(x,y)\frac{dy}{y^{n+1}} \right) \frac{dx}{x^{n+1}},$$

this last contour integral χ having error $|f_{n\mathbf{1}} - \chi| = O(\tau^n)$ for our desired τ, provable via Lemma 5.1 of [14]. This completes Step 2.

We now attempt to bound $|I - I_{loc}|$ and $|I_{out}|$. To bound $|I - I_{loc}|$, first consider

$$J = \left| \frac{1}{2\pi i} \int_{|y|=w_2-\delta} \frac{1}{1 - ax - by + cxy} \cdot \frac{dy}{y^{n+1}} \right|$$

Note that our function $F(x,y) = \frac{1}{1-ax-by+cxy}$ has power series expansion about zero

$$\sum_{k=0}^{\infty} - \frac{(x-b)^k}{(ax-1)^{k+1}} y^k,$$

converging iff $|y| < |g(x)|$. But for any fixed $x \in \mathcal{N}'$, our series converges by our above inequality involving ζ, hence by the 1-D Cauchy integral formula the above integral J is equal to:

$$\left| [y^n] \sum_{k=0}^{\infty} - \frac{(x-b)^k}{(ax-1)^{k+1}} y^k \right| = \frac{|cx-b|^n}{|ax-1|^{n+1}}.$$

And so, for this $x \in \mathcal{N}'$, applying our inequality $\zeta \le |g(x)|$ and noting $|x| \ge w_1$ we obtain:

$$(*) = \frac{|cx-b|^n}{|ax-1|^{n+1}} \le \ldots \le \frac{1}{1-aw_1} \frac{1}{\zeta^n}.$$

Since

$$I - I_{loc} = \frac{1}{(2\pi i)^2} \int_{\mathcal{N}}' \int_{|y|=w_2-\delta} F(x,y) \frac{dydx}{y^{n+1}x^{n+1}},$$

from the Max-modulus-area integral bound we find

$$|I - I_{loc}| \le \frac{1}{1-aw_1} \left(\frac{1}{w_1 \zeta} \right)^n.$$

Next, to bound I_{out}. For $x \in \mathcal{N}$, since $|g(x)| \le \eta < w_2 + \delta = |y|$, it follows that

$$F(x,y) \leq \frac{1}{(w_2+\delta-\eta)(b-cw_1)},$$

and again the max modulus area integral bound implies

$$|I_{out}| \leq \frac{(w_2+\delta)^{-n}(w_1)^{-n}}{(w_2+\delta-\eta)(b-cw_1)}.$$

So, take

$$c_1 := \frac{1}{1-aw_1},$$

$$c_2 := \frac{1}{(w_2+\delta-\eta)(b-cw_1)},$$

$$r_1 := \frac{1}{w_1\zeta},$$

$$r_2 := \frac{1}{w_1(w_2+\delta)}.$$

Note that the c_i are positive and the r_i are less than $w_1 w_2$, our bound for the exponential growth. Since $max(r_1, r_2) = r_1$, by the triangle inequality we obtain the following bound, which holds for any positive integer n:

$$|f_{n\mathbf{1}} - \chi| \leq |I - I_{loc}| + |I_{out}| \leq c_1 r_1^n + c_2 r_2^n \leq (c_1+c_2) r_1^n,$$

and so with $c := c_1 + c_2$, $\tau := r_1$, we have successfully completed Step 3.

Now, onto Step 4, computing $N_f \in \mathbb{Z}_+$ such that for all $n \geq N$, $|\chi - \lambda| \leq \lambda - c\tau^n$. (This won't be easy!) Note that existence of such N_f is guaranteed by the fact that τ^n has smaller exponential growth than λ. (Indeed, $\tau < w_1 w_2$, as we have shown).

From the definition of $\chi \sim \lambda$, we see that it suffices to pick an $\epsilon \in (0,1)$ and then find an N such that for all $n \geq N$, $c\tau^n \leq (1-\epsilon)\lambda$.

We write $\mathcal{N}$ in terms of our parametrization:

$$\mathcal{N} = \{x \in \mathbb{C} : |x| = w_1, Arg(x) \in (-\delta, \delta)\}.$$

Fixing $x \in \mathcal{N}$, by the proof of Corollary 5.1, page 207 of [14], the inner integrand of χ

$$\frac{1}{2\pi i}\left(\int_{|y|=w_2+\delta} F(x,y)\frac{dy}{y^{n+1}} - \int_{|y|=w_2-\delta} F(x,y)\frac{dy}{y^{n+1}}\right)$$

has a unique pole in $w_2 - \delta \le |y| \le w_2 + \delta$ at $g(x)$ hence applying the 1-D Residue Theorem we determine that this integral has value

$$\frac{-1}{b - cx} \cdot \frac{1}{g(x)^{n+1}}.$$

Parametrizing χ with $x = w_1 e^{i\theta}$, one can then check that

$$\chi = \frac{(w_1 w_2)^{-n}}{2\pi} \int_{-\delta}^{\delta} A(\theta) e^{-n\phi(\theta)} d\theta,$$

where

$$A(\theta) := \frac{1}{1 - a w_1 e^{i\theta}}$$

and

$$\phi(\theta) := Log\left[\frac{a w_1 e^{i\theta} - 1}{c w_1 e^{i\theta} - b} \cdot \frac{c w_1 - b}{a w_1 - 1}\right] + i\theta.$$

Let us now verify that A and ϕ are holomorphic on $B(0, 2\delta) \subset \mathbb{C}$, that way we can talk about their (complex) Taylor series. I.e., we now regard the variable θ as complex. Verifying $A(\theta)$ is holomorphic on $B(0, 2\delta)$ simply means checking that the denominator of A is nonzero, but this can be done with a simple application of the reverse triangle inequality, using the known inequalities $Im(\theta) < 2\delta$ (from the ball) and $\delta > ln(\sqrt{a w_1})$ (since we *know* $a w_1 < 1$, and $\delta > 0$).

To ensure $\phi(\theta)$ is holomorphic we need to verify that the argument of Log is not a nonpositive real, equivalent to showing $\frac{a w_1 e^{i\theta} - 1}{c w_1 e^{i\theta} - b}$ isn't. But for this to happen, its imaginary part must be zero, and setting $Im(\frac{a w_1 e^{i\theta} - 1}{c w_1 e^{i\theta} - b}) = 0$ and using Reduce yields an equation in which $Re(\theta)$ must be zero. But when $Re(\theta) = 0$, applying Euler's formula gives

$$\frac{a w_1 e^{i\theta} - 1}{c w_1 e^{i\theta} - b} > \frac{a w_1 e^{-2\delta} - 1}{c w_1 e^{2\delta} - b}.$$

(Note that since the LHS quantity is real, the inequality here actually *does* make sense!) Looking at the numerator of this lower bound, our inequalitiy $\delta > ln(\sqrt{a w_1})$ for δ shows that it is negative, and likewise $\delta < ln(\sqrt{\frac{b}{c w_1}})$ shows that the denominator is negative, thus the ratio is positive for our choice of δ, as needed.

Thus ϕ too is holomorphic on $B(0, 2\delta)$, and hence by a standard result of complex variables, have partial derivativs of all ordres with repect to the real and imaginary parts of our variable, hence regarding θ as real and in $(-2\delta, 2\delta)$ our functions are smooth. Using Taylor's Theorem, we can find expansions for A and ϕ about 0 in a slightly smaller neighborhood in $\mathbb{C}$:

$$\phi(\theta) = \frac{1}{2}\phi''(0)\theta^2 + O(\theta^3), \quad (|\theta| \to 0),$$

with Big-O constants

$$\begin{cases} z_0 := 4\delta/3 \\ c_0 := \frac{8^3}{81} \max_{|\theta|=3\delta/2} |\phi| \end{cases},$$

and

$$A(\theta) = \frac{1}{1 - aw_1} + O(\theta), \quad (|\theta| \to 0)$$

with constants

$$\begin{cases} z_1 := 4\delta/3 \\ c_1 := 8 \max_{|\theta|=3\delta/2} |A| \end{cases}.$$

Note that $\phi''(0)$ is computable purely in terms of a, b, and c, and with a little bit of work one can bound the maxes in our c_i to obtain constants (relabeling our c_i to *be* these new bounds):

$$\begin{cases} z_0 := 4\delta/3 \\ c_0 := \frac{8^3}{81} \left[\max\left\{ln\left(\frac{aw_1 e^{3\delta/2}+1}{|cw1e^{-3\delta/2}-b|}\right), 0\right\} + \max\{ln|1/w_2|, 0\} + \pi + 3\delta/2\right] \end{cases},$$

and

$$\begin{cases} z_1 := 4\delta/3 \\ c_1 := 8 \cdot \frac{1}{1-aw_1 e^{3\delta/2}} \end{cases}.$$

Now, take $B_n := n^{-\alpha}$, where, as you'll recall, α is a rational number in $(1/3, 1/2)$. On our small neighborhood about the origin ($B(0, z_0)$, to be specific), when $B_n \leq |\theta|$, from the prior estimate we obtain

$$-nRe(\phi) \leq \frac{-\phi''(0)}{2} n^{1-2\alpha} + O(1), \quad (n \to \infty)$$

with Big-O constants

$$\begin{cases} N_2 := \lceil \delta^{-1/\alpha)}) \rceil \\ c_2 := c_0 \end{cases},$$

hence, with the same constants,

$$|e^{-n\phi(\theta)}| \leq e^{\frac{-\phi''(0)}{2}n^{1-2\alpha}+O(1)}, \quad (n \to \infty).$$

We can use this small neighborhood to split up our integral as

$$\int_{-\delta}^{\delta} A(\theta)e^{-n\phi(\theta)}d\theta = \int_{-\delta}^{-B_n} A(\theta)e^{-n\phi(\theta)}d\theta + \int_{-B_n}^{B_n} A(\theta)e^{-n\phi(\theta)}d\theta + \int_{B_n}^{\delta} A(\theta)e^{-n\phi(\theta)}d\theta$$

thus by the above and our series for A obtain

$$\int_{-B_n}^{B_n} A(\theta)e^{-n\phi(\theta)}d\theta + O(e^{\frac{-\phi''(0)}{2}n^{1-2\alpha}}), \quad (n \to \infty),$$

with constants

$$\begin{cases} N_3 := N_2 \\ c_3 := 2\delta e^{c_2}(\frac{1}{1-aw_1} + c_1\delta) \end{cases}.$$

Now, for $|\theta| < B_n$, we seek to estimate our integral using a standard Gaussian integral. To do this, first note that by our series expansions for A and ϕ given above, we have $A(\theta) = \frac{1}{1-aw_1} + O(B_n), \quad n \to \infty$ and $e^{-n\phi(\theta)} = e^{-\frac{\phi''(0)}{2}n\theta^2}(1 + O_\mu(nB_n^3)), \quad n \to \infty$ with constants

$$\begin{cases} N_4 := N_3 \\ c_4 := c_1 \end{cases}, \qquad \begin{cases} N_6 := \max(N_2, \lceil \left(\frac{c0}{\mu}\right)^{1/(3\alpha-1)} \rceil) \\ c_6 := e^{\mu}c_0 \end{cases},$$

respectively; note that the parameter μ can be any positive number.

We sub these into our expression for χ, then apply a standard Big-O trick for integrals, obtaining

$$\chi = \frac{(w_1w_2)^{-n}}{2\pi}\left[\frac{1}{1-aw_1}\left(\int_{-B_n}^{B_n} e^{-\frac{\phi''(0)}{2}n\theta^2}d\theta\right)(1 + O(nB_n^3)) + O(e^{-\frac{\phi''(0)}{2}n^{1-2\alpha}})\right].$$

From here, noticing that

$$\int_{-\infty}^{\infty} e^{-\frac{\phi''(0)}{2}n\theta^2}d\theta = \sqrt{\frac{2\pi}{n\phi''(0)}},$$

we can, through a suitable change of variables, split up our integral, insert the above estimate and then "add back the tails" to obtain the following:

$$\int_{-B_n}^{B_n} e^{-\frac{\phi''(0)}{2}n\theta^2} d\theta = \sqrt{\frac{2\pi}{n\phi''(0)}} + O(e^{-\frac{\phi''(0)}{2}n^{1-2\alpha}}), \quad n \to \infty,$$

keeping track of the explicit constants associated with the Big-O terms in a similar fashion as above.

From here, we sub this new estimate into χ, perform several standard Big-O operations, apply Lemma 1 multiple times (setting $\mathcal{H} = \phi''(0)$, an association we shall keep throughout the remainder of the derivation) to combine error terms, and simplify further to obtain a final expression for χ. We keep track of all explicit constants along the way. Since these operations are standard, we omit the details. However, we refer the reader to §5.1 of [14], especially p. 193-194, for a similar example without explicit error bounds.

Anyway, when the dust settles, we have:

$$\chi = \frac{(w_1 w_2)^{-n}}{2\pi} \cdot \frac{1}{1 - aw_1} \cdot \frac{1}{\sqrt{\phi''(0)}}(1 + O(n^{1-3\alpha})), \quad n \to \infty,$$

with Big-O constants

$$\begin{cases} N_{16} := \max(1, \lceil \delta^{-1/\alpha}) \rceil, \left\lceil \left(\frac{c0}{\mu}\right)^{1/(3\alpha-1)} \right\rceil, \tilde{N}) \\ c_{16} := \text{(See table below)} \end{cases},$$

where $\tilde{N}$ is from the Lemma, c_0 is as given above, and c_{16} is gotten by forward-substituting all constants in the following table:

Constant $\mathbf{c_i}$	Value
c_0	$\frac{8^3}{81}\left[\max\left\{ln\left(\frac{aw_1e^{3\delta/2}+1}{\lvert cw1e^{-3\delta/2}-b\rvert}\right),0\right\}+\max\{ln\lvert 1/w_2\rvert,0\}+\pi+3\delta/2\right]$
c_1	$8\cdot\frac{1}{1-aw_1e^{3\delta/2}}$
c_2	c_0
c_3	$2\delta e^{c_2}(\frac{1}{1-aw_1}+c_1\delta)$
c_4	c_1
c_5	c_0
c_6	$e^{\mu}c_5$
c_7	$(1-aw_1)c_4$
c_8	$(1-aw_1)c_6c_4$
c_9	$\frac{1}{2}\sqrt{\frac{2\pi}{\mathcal{H}}}$
c_{10}	$2c_9$
c_{11}	c_9c_{10}
c_{12}	$c_{11}c_{10}^{-1}$
c_{13}	1
c_{14}	$c_9+c_{12}+c_{13}$
c_{15}	$(1-aw_1)c_3\sqrt{\frac{2\pi}{\mathcal{H}}}$
c_{16}	$c_{14}+c_{15}$

By the above estimate, for any $n\in\mathbb{Z}_+$ greater than $N_{17}:=\max(N_{16},\left\lceil\left(\frac{c_{16}}{\epsilon}\right)^{1/(3\alpha-1)}\right\rceil$, we'll have

$$|\chi-\lambda|\leq c_{16}n^{1-3\alpha}\lambda\leq\epsilon\lambda.$$

Finally, we need to find a value $N\in\mathbb{Z}_+$ so that for all greater n, $\epsilon\lambda\leq\lambda-c\tau^n$ with c and τ from the previous step in our "Main Algo." [1] This is the same as finding N such that for all greater n, $c\tau^n\leq(1-\epsilon)\lambda$. Thankfully, we can apply Lemma 2 of the previous section directly to the situation at hand because of the form of λ; we compute the number N as described in the proof of Lemma 2. (Since $d=2$, we fall into Case 3 of the algorithm described in the proof).

This gives us the desired N, and once we have done this, we can now put

$$N_f:=\max(N_{17},N).$$

This is our desired *final index*; expanding and collecting the constants hidden in N_{17} will give us the form as desired in the statement of Computation 1,

[1]See §2.1

and clearly this satisfies the requirement for our N_f since for any n larger than N_f, we have

$$|f_{n\mathbf{1}} - \lambda| \le |f_{n\mathbf{1}} - \chi| + |\chi - \lambda| < \lambda,$$

which in particular implies that for any $n \ge N_f$, $f_{n\mathbf{1}} > 0$. □

4.3 Derivation of GRZ Bound

Proof. As before, we run all four steps in our "Main Algorithm" from Section 2. We shall use fewer words than last time.

Starting with $d \ge 4$ a positive integer,

$$F(\boldsymbol{z}) = F_{d!,d}(\boldsymbol{z}) = \frac{1}{H(\boldsymbol{z})}, \qquad H(\boldsymbol{z}) = 1 - \sum_{i=1}^{d} z_i + d! \prod_{i=1}^{d} z_i,$$

which is holomorphic (hence analytic) in a neighborhood of the origin, we first verify that F satisfies the hypotheses in Step 1 of the Main Algo.

To do this, first observe the following general facts:

Lemma 3.

$$\forall d \ge 4, d! < (d-1)^{d-1},$$

which is provable by induction, and its immediate consequence

Lemma 4.

$$\forall d \ge 4, \quad \forall \rho \in (\frac{1}{d}, \frac{1}{d-1}), \quad d!\rho^{d-1} < 1,$$

Note that by Prop. 4.2 of [5], $\mathcal{V} = \mathcal{V}(F)$ is smooth, and by Thm. 1.9 of that same paper, the coefficinets of $diag(F)$ are eventually positive. To find a stricly minimal smooth-contributing nondegenerate point of $\mathcal{V}$, note that by their Lemma 2.3 of [5], minimum modulus zeros of $H(x, x, \dots, x) = 1 - dx + d!x^d$ are (strictly) minimal smooth critical points of F. Also, by Prop. 4.3, for $d \ge 4$, $1 - dx + d!x^d$ has a unique root $\rho_d = \rho \in [1/d, 1/(d-1))$.

(Note that for $d = 4$, ρ is exactly solvable in terms of radicals. It is given by:

$$\rho_4 = \frac{\sqrt{3(1+2^{1/3})} - \sqrt{3(-1-2^{1/3}+2\sqrt{3/(1+2^{1/3})})}}{6 \cdot 2^{1/3}}.$$

For higher values of d ρ is not solvable by radicals.

Anyways, by Lemma 2.3 the point $\boldsymbol{\rho} \coloneqq (\rho, \ldots, \rho)$ is strictly minimal smooth critical, hence contributing by Prop. 3.6 of [9].

Finally, let us show that $\boldsymbol{\rho}$ is nondegenerate, thereby completing the checks of Step 1. Proving $\boldsymbol{\rho}$ nondegenerate simply means showing the Hessian matrix $\mathcal{H}$ of a certain function ϕ (introduced later) associated with F evaluated at the origin has nonzero determinant. Alternatively, one can use Lemma 5.5 of [14] to compute $\mathcal{H}$ directly, which is what we do. Its form is relatively simple, and using some standard results on the determinants (See, e.g., [21] 3.1 and 3.3), one can show that for $d \geq 4$, with $U \coloneqq \frac{d!\rho^{d-1}}{-1+d!\rho^{d-1}}$ (which is negative by our Lemma 4)

$$\det \mathcal{H} = d \cdot (1 - U)^{d-1},$$

which is positive, hence nonzero. Thus, $\boldsymbol{\rho}$ is indeed nondegenerate, meaning we have completed Step 1 of the Main Algo.

At this point we can compute our "out-of-the-box" asymptotic estimate using Theorem 5.2 of [14], obtaining $\lambda(n)$.

Now we introduce the usual objects associated with our analysis:

- $\mathcal{T} \coloneqq T(\hat{\boldsymbol{\rho}})$, the $(d-1)$-dimensional polytorus centered at $\hat{\boldsymbol{\rho}} \in \mathbb{C}^{d-1}$,
- $\delta \coloneqq \frac{1}{(d-1)^{d-1}}$, a positive constant
- $\mathcal{N} = \{(\rho e^{i\theta_1}, \ldots, \rho e^{i\theta_{d-1}}) : \theta_1, \ldots, \theta_{d-1} \in (-\delta, \delta)\}$, our neighborhood of $\hat{\boldsymbol{\rho}}$ in $\mathcal{T}$,
- $\mathcal{N}' \coloneqq \mathcal{T} \setminus \mathcal{N}$, and
- Our analytic parametrization $z_d = w = g(\hat{\boldsymbol{z}})$ of $\mathcal{V}$ for $\hat{\boldsymbol{z}} \in \mathcal{N}$ defined by:

$$g(\hat{\boldsymbol{z}}) \coloneqq \frac{1 - z_1 - \cdots - z_{d-1}}{1 - d! z_1 \ldots z_{d-1}}.$$

It is easy to verify g is holomorphic, hence analytic, using Lemmas 3 and 4.

Next one shows that our objects satisfy requirements i.) - iii.) on page 206 of [14], thereby enabling us to perform an asymptotic analysis using saddle point integration. Proving i.) and iii.) is similar to the bivariate case, hence is omitted. As with the bivariate case, instead of proving ii.) we prove something slightly stronger, namely:

1. $\exists \eta \in \mathbb{R}_+$ such that $\forall \hat{\boldsymbol{z}} \in \mathcal{N}, w - \delta < w \leq |g(\hat{\boldsymbol{z}})| \leq \eta < w + \delta$, and

2. $\exists \zeta \in \mathbb{R}_+$ such that $\forall \hat{\boldsymbol{z}} \in \mathcal{N}', w - \delta < w < \zeta \leq |g(\hat{\boldsymbol{z}})|$.

We first rewrite $|g| : \mathcal{N} \to \mathbb{R}$ in terms of our parmetrization of $\mathcal{N}$. One can then show through simple inequalities involving cosine and our choice of δ that

$$\eta := \sqrt{\frac{(1 - \rho(d-1)cos\delta)^2 + \rho^2((d-1)sin\delta)^2}{(1 - d!\rho^{d-1})^2}}$$

is a uniform bound satisfying 1.

As for 2., since both g and our parametrization of $\mathcal{T}$ are continuous, $\mathcal{N}'$ compact, $|g|$ must obtain a minimum on $\mathcal{N}'$, and it is apparent that this minimum lies on the boundary of $\mathcal{N}'$. We would like to conclude that

$$\zeta := \min_{\hat{\boldsymbol{z}} \in \mathcal{N}'} |g(\hat{\boldsymbol{z}})|$$

is a value satisfying 2., and a somewhat systematic battery of numerical trials for low values of d seems to suggest this. Unfortunately, the author admits that he does not, at this time, know how to prove that this ζ satisfies the desired inequalities, nor how to find a nice expression for $\min_{\hat{\boldsymbol{z}} \in \mathcal{N}'} |g(\hat{\boldsymbol{z}})|$, or whether one even exists.

We leave this as:

Conjecture 1.

$$\rho + \delta > \min_{\hat{\boldsymbol{z}} \in \mathcal{N}'} |g(\hat{\boldsymbol{z}})| > \rho = |g(0, 0, \ldots, 0)|,$$

with $\zeta = \min_{\hat{\boldsymbol{z}} \in \mathcal{N}'} |g(\hat{\boldsymbol{z}})|$ *being expressible in terms of elementary functions.*

... and since the numerics strongly suggest it, we shall regard the conjecture as true for the remainder of the derivation. Assuming that Conejcture 1 holds, clearly ii.) follows from 1. and 2., hence we are done.

From here, we introduce our usual Cauchy integral I for our diagonal coefficients $f_{n\mathbf{1}}$, along with the associated integrals I_{loc}, I_{out}, and χ. (See last section). In this case, these integrals are:

$$f_{n\mathbf{1}} = I := \frac{1}{(2\pi i)^d} \int_{\mathcal{T}} \left(\int_{|z_d| = \rho - \delta} F_{d!,d}(\boldsymbol{z}) \frac{dz_d}{z_d^{n+1}} \right) \frac{d\hat{\boldsymbol{z}}}{\hat{\boldsymbol{z}}^{(n+1)\hat{\mathbf{1}}}},$$

$$I_{\text{loc}} := \frac{1}{(2\pi i)^d} \int_{\mathcal{N}} \left(\int_{|z_d| = \rho - \delta} F_{d!,d}(\boldsymbol{z}) \frac{dz_d}{z_d^{n+1}} \right) \frac{d\hat{\boldsymbol{z}}}{\hat{\boldsymbol{z}}^{(n+1)\hat{\mathbf{1}}}},$$

$$I_{\text{out}} := \frac{1}{(2\pi i)^d} \int_{\mathcal{N}} \left(\int_{|z_d|=\rho+\delta} F_{d!,d}(\boldsymbol{z}) \frac{dz_d}{z_d^{n+1}} \right) \frac{d\hat{\boldsymbol{z}}}{\hat{\boldsymbol{z}}^{(n+1)\hat{\mathbf{1}}}}, \text{ and}$$

$$\chi := I_{\text{loc}} - I_{\text{out}} = \frac{-1}{(2\pi i)^d} \int_{\mathcal{N}} \left(\int_{|z_d|=\rho+\delta} F_{d!,d}(\boldsymbol{z}) \frac{dz_d}{z_d^{n+1}} - \int_{|z_d|=\rho-\delta} F_{d!,d}(\boldsymbol{z}) \frac{dz_d}{z_d^{n+1}} \right) \frac{d\hat{\boldsymbol{z}}}{\hat{\boldsymbol{z}}^{(n+1)\hat{\mathbf{1}}}},$$

Noting that a power series expansion for $F_{d!,d}$ about 0 is

$$F(\boldsymbol{z}) = \sum_{\ell=0}^{\infty} \frac{(1 - d! \prod_1^{d-1} z_j)^\ell}{(1 - \sum_1^{d-1} z_j)^{\ell+1}} z_d^\ell,$$

converging iff $|z_d| < |g(\hat{\boldsymbol{z}})|$, we use a procedure completely analogous to that in the bivariate case to bound $I - I_{loc}$ and I_{out}. (To bound $I - I_{loc}$, use our power series expansion and the Cauchy integral formula on the innermost integral, followed by an application of the max modulus area bound. For I_out, one needs only max mod area. These arguments are routine, we omit the details and instead refer the reader to the previous section or the proof of Lemma 5.1 of [14]).

We end up with:

$$|I - I_{loc}| \leq c_1 r_1^n,$$

$$|I_{out}| \leq c_2 r_2^n,$$

where

$$\begin{cases} c_1 := \frac{1}{1-(d-1)\rho}, & r_1 := \frac{1}{\rho^{d-1}\zeta} \\ c_2 := \frac{1}{(\rho+\delta-\eta)(1-d!\rho^{d-1})}, & r_2 := \frac{1}{\rho^{d-1}(\rho+\delta)} \end{cases},$$

the c_i's *indeed* being positive and the r_i's positive and less than the exponential growth bound $|w_1 \ldots w_d|^-1$. Thus putting $c := c_1 + c_2$, $\tau := max(r_1, r_2) = r_1$, we have for any $n \in \mathbb{Z}_+$ $|f_{n\mathbf{1}} - \chi| \leq c\tau^n$, completing Step 3.

As for Step 4, *again*, we need to find some index so that for all larger n, $|\chi - \lambda| \leq \lambda - c\tau^n$. But $\chi \sim \lambda$ implies that for any choice of $\epsilon \in (0, 1)$, we can instead find an index so that for larger n, $\epsilon\lambda \leq \lambda - c\tau^n$.

To do this, we carry out the same general procedure as for the Bivariate class.

First we note that $\chi = I_{loc} - I_{out}$ has, for each $\hat{\boldsymbol{z}} \in \mathcal{N}$, a unique pole between $|w| = \rho - \delta$ and $|w| = \rho + \delta$, located at $g(\hat{\boldsymbol{z}})$, thus by our familiar one-dimensional

Residue Theorem we have

$$\chi = \frac{1}{(2\pi i)^{d-1}} \int_{\mathcal{N}} \frac{(1-d!z_1 \dots z_{d-1})^n}{z_1^{n+1} \dots z_{d-1}^{n+1}(1-z_1-\dots-z_{d-1})^{n+1}} d\hat{\boldsymbol{z}}.$$

With our parametrization of $\mathcal{N}$ this becomes

$$\chi = \frac{\rho^{-dn}}{(2\pi)^{d-1}} \int_{-\delta}^{\delta} \int_{-\delta}^{\delta} \cdots \int_{-\delta}^{\delta} A_n(\theta_1, \dots \theta_{d-1}) e^{-n\phi_n(\theta_1,\dots,\theta_{d-1})} d\boldsymbol{\theta},$$

where

$$A_n(\boldsymbol{\theta}) = \frac{1}{1-\rho(e^{i\theta_1}+\dots+e^{i\theta_{d-1}})},$$

$$\phi_n(\boldsymbol{\theta}) = Log\left[\frac{1-\rho(e^{i\theta_1}+\dots+e^{i\theta_{d-1}})}{1-d!\rho^{d-1}e^{i(\theta_1+\dots+\theta_{d-1})}} \cdot \frac{1}{\rho}\right] + i\theta_1 + \dots + i\theta_{d-1}.$$

One can easily show that A_n is holomorphic on $B(0,2\delta)^{d-1} \subset \mathbb{C}^{d-1}$, hence $\mathbb{C}$-analytic, thus each of its real and imaginary parts will be analytic.

To show that ϕ_n is holomorphic on $B(0,2\delta)^{d-1} \subset \mathbb{C}^{d-1}$, one can first show that the image of $B(0,2\delta)^{d-1}$ under our parametrization lies in $B(\rho, \rho|e^{2\delta}-1|)^{d-1})$. This involves a simple application of the Maximum Principle from complex analysis, needed to show that

$$\max_{\theta \in B(0,2\delta))} |e^{i\theta}-1| = |e^{2\delta}-1|.$$

Next one can show that the function $f : (B(\rho, \rho|e^{2\delta}-1|)^{d-1}))^{d-1} \to \mathbb{C}$ defined by

$$f(\hat{\boldsymbol{\theta}}) := \frac{(1-d!\rho^{d-1})^n}{\rho^{(d-1)(n+1)}e^{i(n+1)(\theta_1+\dots+\theta_{d-1})}(1-\rho(e^{i\theta_1}+\dots+e^{i\theta_{d-1}}))}$$

is holomorphic in $\mathbb{C}^{d-1}$, through showing the denominator has positive magnitude at each point in our domain. Note that we need to use the fact that δ is so small that $\frac{1}{\rho(d-1)} - 1 > |e^{2\delta}-1|$ in our proof of this; such a δ *can* be chosen algorithmically by choosing smaller and smaller values of δ.

Finally, composing f with our parametrization yields the integrand of χ; since we already know A_n is holomoprhic and nonzero everywhere in our domain, dividing by A_n, we conclude that ϕ_n is too.

Thus, by a result of Hartog's from several complex variables [22], ϕ_n is holomorphic in each variable, so is, in particular, differentiable as a function restricted to $(-2\delta, 2\delta)^{d-1}$.

Next, we find the explicit error bounds associated with the Taylor Series expansion for $\phi := \phi_n$

$$\phi(\hat{\boldsymbol{\theta}}) = \frac{1}{2}\hat{\boldsymbol{\theta}}^{\intercal}\mathcal{H}\hat{\boldsymbol{\theta}} + O(\|\hat{\boldsymbol{\theta}}\|_1^3), \quad \|\hat{\boldsymbol{\theta}}\|_1 \to 0.$$

Doing this involves bounding the maximum of the set

$$\max\left\{\partial^{(r_1,r_2,\ldots,r_{d-1})}\phi(\hat{\boldsymbol{\theta}}) : \hat{\boldsymbol{\theta}} \in \left[-\frac{3\delta}{2}, \frac{3\delta}{2}\right]^{d-1} \wedge r_1,\ldots,r_{d-1} \in \mathbb{N} \wedge \sum_{i=1}^{d-1} r_i = 3\right\},$$

which does exist. Here the multi-indexed partial derivative notation (along with some other notation, seen below) is borrowed from [23]. An upper bound for the set above, which we'll call M, can be obtained algorithmically by:

1. For each of the multiderivatives in the above set, compute an upper bound for its magnitude on $[-3\delta/2, 3\delta/2]^{d-1}$ by doing the following:
   - Take the multi-derivative of ϕ. This derivative is guaranteed to exist on all of $[-3\delta/2, 3\delta/2]^{d-1}$, and will look like

     $$1/(\text{argument of the Log}) \cdot (\text{some polymomial in } \theta)$$

   - Apply the triangle inequality repeatedly to form the "trivial" upper bound of this.
2. Maximize over the set of these bounds to obtain an upper bound for the set above. DONE.

From this bound M and Taylor's Theorem, one can derive the basic estimate

$$\phi(\hat{\boldsymbol{\theta}}) = \sum_{|\omega|\le 2} \frac{\partial\omega\phi(\hat{\boldsymbol{\theta}})}{\omega!}\hat{\boldsymbol{\theta}}^{\omega} + O(\|\hat{\boldsymbol{\theta}}\|_\infty^3), \qquad \|\hat{\boldsymbol{\theta}}\|_\infty \to 0,$$

from which our desired estimate follows. The constants associated with the O-term are

$$\begin{cases} z_1 := \frac{3\delta}{2} \\ c_1 := (d-1)^3 M \end{cases}.$$

Thus, we have

$$\phi(\hat{\boldsymbol{\theta}}) = \frac{1}{2}\hat{\boldsymbol{\theta}}^{\top}\mathcal{H}\hat{\boldsymbol{\theta}} + O(\|\hat{\boldsymbol{\theta}}\|_1^3), \quad \|\hat{\boldsymbol{\theta}}\|_1 \to 0.$$

Now, the Morse Lemma (Lemma 5.3.1, [12]) guarantees the existence of a biholomorphic change of variables which will convert our integral χ into an equivalent one having *standard phase* in the exponential– i.e., so the exponential $e^{-\frac{n}{2}\hat{\boldsymbol{\theta}}^{\top}\mathcal{H}}$ gets converted into one of the form $e^{y_1^2+\cdots+y_{d-1}^2}$. While 5.3.1 gives us a way of computing the change of variables constructively, it is a bit difficult to work with. Instead, we shall pursue the alternate route of just isolating the "standard part" of our exponential argument. I.e., we seek to estimate the cross-terms by a constant depending on n. This seems to be a more natural generalization of the procedure carried out in the bivariate class example, anyhow.

So, in analogy to the bivariate example, we choose $\alpha \in (1/3, 1/2) \cap \mathbb{Q}$, put $B_n := n^{-\alpha}$ and split up our domain of integration $[-\delta,\delta]^{d-1} \subseteq \mathbb{R}^{d-1}$ into $(d-1)^{d-1}$ subregions:

- One where $\|\hat{\boldsymbol{\theta}}\|_\infty \le B_n$
- The other $(d-1)^{d-1}-1$ such that for some $i \in [d-1]$, $|\theta_i| \ge B_n$.

...and attempt to estimate our integral on each subregion.

The integrals over the $(d-1)^{d-1}-1$ *non-central* regions (i.e. those where $|theta_i| \ge B_n$ for some i) will contribute an error term of $O(e^{-\frac{H_{1,1}}{2}n^{1-2\alpha}}) \quad n \to \infty$ with constants N_4, c_4 (SEE TABLE BELOW.)

Thus, we have

$$\int_{[-\delta,\delta]^{d-1}} A_n e^{-n\phi_n} = \int_{[-B_n,B_n]^{d-1}} A_n e^{-n\phi_n} + O(e^{-\frac{H_{1,1}}{2}n^{1-2\alpha}}).$$

For the integral over the *central* region $[B_n, B_n]^{d-1}$ (i.e. where $\|\hat{\boldsymbol{\theta}}\|_\infty \le B_n$), one can look at a second order Taylor series expansio for A_n and compute an upper bound for

$$\max\left\{|\partial_{\theta_j} A(\hat{\boldsymbol{\theta}})| : \hat{\boldsymbol{\theta}} \in [-3\delta/2, 3\delta/2]^{d-1}, j \in [d-1]\right\},$$

namely

$$\tilde{M} := \frac{\rho}{(1-(d-1)\rho)^2},$$

to obtain

$$A_n = \frac{1}{1-(d-1)\rho} + O(B_n), \qquad n \to \infty$$

with constants N_5, c_5 (see Table), along with another easy estimate for $e^{-n\phi}$ on this region via Taylor series:

$$e^{-n\phi(\hat{\boldsymbol{\theta}})} = e^{-\frac{n}{2}\hat{\boldsymbol{\theta}}^\top \mathcal{H}\hat{\boldsymbol{\theta}}}(1 + O(n^{1-2\alpha}))$$

with constants N_7, c_7 (Table.)

Now, we sub our estimates into the integral $\int_{[-B_n,B_n]^{d-1}} Ae^{-n\phi}$ and use standard Big-O tricks to simplify to

$$\int_{[-B_n,B_n]^{d-1}} Ae^{-n\phi} = \frac{1}{1-(d-1)\rho}\int_{[-B_n,B_n]^{d-1}} e^{-\frac{n}{2}\hat{\boldsymbol{\theta}}^\top \mathcal{H}\hat{\boldsymbol{\theta}}}d\hat{\boldsymbol{\theta}}(1 + O(n^{1-3\alpha})).$$

It remains to estimate the integral

$$I := \int_{[-B_n,B_n]^{d-1}} e^{-\frac{n}{2}\hat{\boldsymbol{\theta}}^\top \mathcal{H}\hat{\boldsymbol{\theta}}}d\hat{\boldsymbol{\theta}}.$$

As we alluded to, we will do this by making a change of variables in order to eliminate the cross terms in the quadratic form $\hat{\boldsymbol{\theta}}^\top \mathcal{H}\hat{\boldsymbol{\theta}}$. To avoid mixing notations and increase clarity, let $\boldsymbol{x} = \hat{\boldsymbol{\theta}}$ for a moment. With $f(\boldsymbol{x}) = \boldsymbol{x}^\top \mathcal{H}\boldsymbol{x}$, by a theorem of linear algebra, there exists an orthogonal matrix Q so that

$$Q^\top \mathcal{H} Q = \begin{bmatrix} \lambda_1 & 0\dots & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & \dots & 0 \\ \vdots & & \ddots & & \\ 0 & \dots & 0 & 0 & \lambda_{d-1} \end{bmatrix},$$

a diagonal matrix with eigenvalues $\lambda_1, \dots, \lambda_{d-1}$. This, in particular, implies the Principal Axes Theorem, which says that we can find a change of variables $\boldsymbol{y} := Q^{-1}\boldsymbol{x}$ so that

$$f(\boldsymbol{x}) = \boldsymbol{y}^\top \mathcal{H}\boldsymbol{y} = \lambda_1 y_1^2 + \dots + \lambda_{d-1} y_{d-1}^2$$

...i.e. one eliminating cross terms! Let us apply this to our case– the goal is to find an orthogonal matrix Q satisfying the above requirement. To do this, one can first show directly that the eigenvalues of $\mathcal{H}$ are

$$\begin{cases} \lambda_1 = \lambda_2 = \cdots = \lambda_{d-2} = 1 - U = \frac{1}{1-d!\rho^{d-1}} > 0 \\ \lambda_{d-1} = \frac{d-(d-1)d!\rho^{d-1}}{1-d!\rho^{d-1}} > 0 \end{cases},$$

and by a direct comparison $\lambda_{d-1} > \lambda_1 = \cdots = \lambda_{d-2}$. From this we find an eigendecomposition for $\mathcal{H}$ and must now turn this into an orthonormal basis for the space spanned by our eigenvectors $v_1, ..., v_{d-1}$. We have one eigenspace of dimension $d-2$, namely $span(v_1, ..., v_{d-2})$; to find an orthonormal basis for this subspace, apply the Gram-Schmidt algorithm and obtain orthonormal vectors $\hat{v_1}, \ldots, \hat{v_{d-2}}$ from $v_1, \ldots, v_{d-2}$. Normalize v_{d-1} to obtain our full set of orthonormal vectors, hence the orthogonal matrix

$$Q := \begin{bmatrix} | & | & & | \\ \hat{v}_1 & \hat{v}_2 & \ldots & \hat{v}_{d-1} \\ | & | & & | \end{bmatrix}$$

such that

$$Q^\top \mathcal{H} Q = D := diag(\lambda_1, \ldots, \lambda_{d-1})$$

and our desired change of variables $\boldsymbol{y} := Q^\top \hat{\boldsymbol{\theta}}$. (So $\hat{\boldsymbol{\theta}} = Q\boldsymbol{y}$.) Applying this transformation and noting that its Jacobian has determinant 1 (since Q is orthogonal), the integral I becomes

$$I = \int_{Q^\top([-B_n, B_n]^{d-1})} e^{-\frac{n}{2}(\lambda_1 y_1^2 + \cdots + \lambda_{d-1} y_{d-1}^2)} d\boldsymbol{y}.$$

As before, we use the standard Gaussian integral

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-\frac{n}{2}(\lambda_1 y_1^2 + \cdots + \lambda_{d-1} y_{d-1}^2)} = \sqrt{\frac{(2\pi)^{d-1}}{n^{d-1}\lambda_1 \ldots \lambda_{d-1}}}$$

and then try to estimate the integral over the region *outside* I's new domain of integration, $Q^\top([-B_n, B_n]^{d-1})$. We note that $Q^\top([-B_n, B_n]^{d-1})$ contains a cuboid of radius $B_n/\sqrt{d-1}$, and then observe that the integral

$$J := \int_{\mathbb{R}^{d-1} - [-B_n/\sqrt{d-1}, B_n/\sqrt{d-1}]^{d-1}} A_n(\boldsymbol{y}) e^{-n\phi_n(\boldsymbol{y})} d\boldsymbol{y}$$

is much easier to estimate; indeed, noting that $\lambda_{d-1} > \lambda_1 = \cdots = \lambda_{d-2}$ and performing a u-substitution, we find an upper bound for J:

$$J \le \frac{(d-1)^{d-1}-1}{2} e^{-\frac{\lambda_1}{2(d-1)} n^{1-2\alpha}}, \qquad \forall n \ge 3.$$

Subbing back into χ and adding back the tails, we obtain

$$\chi = \frac{\rho^{-dn}}{(2\pi)^{d-1}} \int_{[-\delta,\delta]^{d-1}} A_n e^{-n\phi_n} = \frac{\rho^{-dn}}{(2\pi)^{d-1}} \left[\frac{1}{1-(d-1)\rho} J(1+O(n^{1-3\alpha})) + O(e^{-\frac{\mathcal{H}_{1,1}}{2} n^{1-2\alpha}}) \right].$$

Since

$$J = \int_{\mathbb{R}^{d-1}} (\dots) - \int_{\mathbb{R}^{d-1} - Q^\top([-B_n,B_n]^{d-1})} (\dots),$$

and the integral over $\mathbb{R}^{d-1} - Q^\top([-B_n, B_n]^{d-1})$ is less than the integral over $\mathbb{R}^{d-1} - [-B_n/\sqrt{d-1}, B_n/\sqrt{d-1}]^{d-1}$, we find

$$\chi = \frac{\rho^{-dn}}{(2\pi)^{d-1}} \left[\frac{1}{1-(d-1)\rho} \left(\sqrt{\frac{(2\pi)^{d-1}}{n^{d-1}\lambda_1 \dots \lambda_{d-1}}} + O\left(e^{-\frac{\lambda_1}{2(d-1)} n^{1-2\alpha}}\right) \right) (1+O(n^{1-3\alpha})) + O\left(e^{-\frac{\mathcal{H}_{1,1}}{2} n^{1-2\alpha}}\right) \right],$$

where, during the course of our analysis, we have kept track of the constants associated with each error term in the above expression.

Simplifying further, and applying Lemma 1 (with $\mathcal{H}$ set to $-\frac{\lambda_1}{d-1}$) to obtain the constant $\tilde{N} \in \mathbb{N}$ in the same fashion as in the bivariate case we get, at long last, our desired asymptotic:

$$\chi = \frac{\rho^{-dn}}{(2\pi)^{d-1}} \frac{1}{1-(d-1)\rho} \sqrt{\frac{(2\pi)^{d-1}}{n^{d-1}\lambda_1 \dots \lambda_{d-1}}} (1+O(n^{1-3\alpha})), \quad n \to \infty$$

with Big-O constants N_{18}, c_{18} obtained by collecting all the values in the following table:

Index i	**Value** N_i	**Value** c_i
1	$z_1 := 3\delta/2$	$(d-1)^3 M$
2	N_2 (unknown)	c_2 (unknown)
3	N_2	$\frac{(2\delta)^{d-1} e^{c_2}}{1-(d-1)\rho e^{\delta}}$
4	N_3	$((d-1)^{d-1} - 1)c_3$
5	$\left\lceil \left(\frac{1}{\delta}\right)^{1/\alpha} \right\rceil$	$(d-1)\tilde{M}$
6	N_5	$(d-1)^{d-1} M$
7	$\max(N_5, \lceil \tilde{\epsilon}^{1/(1-3\alpha)} \rceil)$	$e^{\tilde{\epsilon}} c_6$
8	N_7	$(1-(d-1)\rho)c_7$
9	$\max(N_5, N_7)$	$c_5 c_7$
10	N_9	$(1-(d-1)\rho)c_9$
11	$\max(N_7, N_8, N_{10})$	$c_7 + c_8 + c_{10}$
12	3	$\frac{(d-1)^{d-1}}{2}$
13	$N_1 2$	$\frac{c_{12}}{1-(d-1)\rho}$
14	$\max(N_{12}, N_{11})$	$c_{13} c_{11}$
15	$\max(N_{13}, N_{14})$	$c_{13} + c_{14}$
16	$\max(N_{15}, N_4)$	$c_{15} + c_4$
17	$\max(N_{16}, \tilde{N})$	c_{16}
18	$\max(N_{17}, N_{11})$	$c_{17} + c_{11}$

This completes Step 4.

Finally, we apply Lemma 2 (with c, τ our c and τ from Step 3, and D the positive constant factor in the definition of λ) to obtain $N \in \mathbb{Z}_+$ such that for all larger n, $\epsilon\lambda < \lambda - c\tau^n$, and note that for any $n \geq N_f := \max(N_{18}, N, \left\lceil \left(\frac{c_{18}}{\epsilon}\right)^{1/(3\alpha-1)} \right\rceil)$, $|\chi - \lambda| < \lambda - c\tau^n$, as needed. Finally, we have our N_f. $\square$

As we shall see in the next section, the values of N_f that we obtain from this method are not very practical, even for small values of d. However, there are some easy optimizations that can be made that could possibly improve our results to the point of being practical to compute on modern machines.

CHAPTER 5

CONCLUSION

5.1 Computer Analysis

At this point we've carried out our analysis on a couple of examples. Our methodology, at least in theory, seems to work. But one question remains: is it practical, for a function $F(\boldsymbol{z}) = \sum_{\boldsymbol{i}\in\mathbb{N}^d} f_{\boldsymbol{i}}\boldsymbol{z}^{\boldsymbol{i}}$, to check $f_{n\mathbf{1}}$ for $n = 1, 2, \ldots, N_f$ for positivity, assuming the existence of an N_f?

As we shall see below, this is highly dependent on both the value d and the function F. For some choices of F it is *certainly* feasible to use our method to decide positivity of F, even on machines with modest resources running un-optimized software. However, we suspect that in general it is not –and probably never will be– feasible to do this for arbitrary d positive and F.

There are some small things we can do to make our N_f smaller, and more likely to be "within the reach" of modern computers; see 5.1.3. But in order for this approach to work on other classes of functions, an overhaul of our approach may be necessary. We will investigate this further in upcoming research.

5.1.1 Bivariate case

We consider, as an example, the case when $a = 3, b = 4, c = 3$.

The first few entries in $diag(F)$'s coefficient series are

$$1, 21, 667, 22869, 836001, \ldots$$

and running our procedure above on this F with parameter values

- $\delta = \frac{1}{2}\min(w_2, \pi/2, \log(\sqrt{\frac{b}{cw_1}})) = w_2/2$
- $\epsilon = 1/2$
- $\alpha = 2/5$
- $\mu = 1/2$

yields an index of $\mathbf{N_f = 1307}$.

For a different choice of parameter values, namely:

- $\delta =$ **(same)**
- $\epsilon = 2/5$

- $\alpha = 639/1280$
- $\mu = 1/2$

we can improve this bound slightly to $\mathbf{N_f} = 1269$. Positivity for all terms up to 1269 can be checked on our machine (Intel i5-9400, 6 cores, 12GB RAM, no SSD), in less than 20 minutes, using Mathematica's default Series expansion routine with no optimizations whatsoever.

Further experimentation with other values of a, b, c yields values for N_f with similar computation times.

5.1.2 GRZ case

Here, even for $d = 4$, it appears that the values of N_f we obtain are prohibitively large. Indeed, with a very reasonable choice of parameters, we have:

$$N_f \geq 4547349426.$$

Computing a Taylor series expansion of this order is certainly impossible using modern machines, and even for a more informed choice of parameters (see below), our bound doesn't improve significantly.

It is worth noting that the 'problematic' step lies in the computation of $\tilde{N}$ from Lemma 2; up to that point, our values of N_i are fairly reasonable (< 1000). This makes sense, as our α results in a very slow rate of decay for $e^{-\frac{\mathcal{H}}{2}n^{1-2\alpha}}$, hence it would take awhile for it to be less than $n^{\frac{1}{2}-3\alpha}$.

Since it is impractical for the (in some sense) 'simplest' example of multivariate rational functions of interest in the literature, one should not expect our N_f's to be small enough in general.

5.1.3 Improvements?

Over the course of our experiments, we noticed that N_f seems to approach a limiting value as α approaches $1/2$ from below. This comes at the expense of other constituent constants appearing in N_f; similarly, ϵ seems to reach a value resulting in optimal N_f at some point bounded away from *both* 0 and 1. We would like to conduct a systematic investigation of the tradeoffs between different choices of parameters, hopefully determining what the limiting value of N_f is as we approach an optimum point in parameter space.

One can also observe that many of the choices we made during our analysis were *not* optimal, by any stretch of the imagination. For instance, the value we obtained for c_0 in the bivariate case was

$$c_0 = \frac{8^3}{81} \max_{|\theta|=3\delta/2} |\phi(\theta)|.$$

Since by the triangle inequality and definition of ϕ we have

$$|\phi| \leq \ln(|g(w_1 e^{i\theta})|/g(w_1)) + \mathrm{Arg}(|g(w_1 e^{i\theta})|/g(w_1)) + |\theta|$$

when $|\theta| = 3\delta/2$, we made the choice to bound the Arg term above by π. This is, to be blunt, "not great," and leads to a faster-growing c_0 and thus a larger N_f; it should not be difficult to find a bound for the Arg term depending on a, b, c and δ, ideally so that the total bound is less than 1. Similar improvements can be made throughout our analysis.

5.2 Future Work

Besides the quick fixes mentioned above, we hope to investigate the following research directions:

- *Exploring other Functions:* Will a similar procedure apply to any other examples of interest in the literature? Can we use existing results on positivity / asymptotic positivity (much as we did here for the GRZ case) to do a majority of the heavy-lifting for Steps 1 and 2? If not, how can we prove the hypotheses of our Main Algo for these functions?

- *Automation:* Many of the steps in our analysis, especially in Steps 3 and 4, seem to be automatable in some generality, as [10] would indicate. Implementing this automation and exploring its limits is an ongoing goal of our reseach. For steps that *do* require some human intervention or "choices," can we find a good set of heuristics for making said choices?

- *Degenerate Cases:* As an alternative to the case when we have eventual positivity, I'd like to explore the feasibility of finding N_f in degenerate cases, such as when our coefficients are eventually zero in a direction. For instance, can we find N_f such that all higher-index coefficients are zero when Proposition 5.6 of [**?**] applies? (This would round out our Algorithm a bit).

- *Poles on Hyperplane Arrangements, and beyond:* Finally, I would like to extend our procedure to include *nonsmooth* cases; as with anything, this would mean first working with specific examples, then attempting to generalize.

Overall, we have investigated the possibility of proving positivity of multivariate rational functions using ACSV and have found a set of principles with the potential for allowing us to automate such proofs in certain cases.

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